Twisted cohomologies of wrap groups over quaternions and octonions.

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Abstract

This article is devoted to the investigation of wrap groups of connected fiber bundles over the fields of real **R**, complex **C** numbers, the quaternion skew field **H** and the octonion algebra **O**. Cohomologies of wrap groups and their structure are investigated. Sheaves of wrap groups are constructed and studied. Moreover, twisted cohomologies and sheaves over quaternions and octonions are investigated as well.

1 Introduction.

Geometric loop groups of circles were first introduced by Lefshetz in 1930-th and then their construction was reconsidered by Milnor in 1950-th. Lefshetz has used the C^0-uniformity on families of continuous mappings, which led to the necessity of combining his construction with the structure of a free group with the help of words. Later on Milnor has used the Sobolev's H^1-uniformity, that permitted to introduce group structure more naturally [36].

The construction of Lefshetz is very restrictive, because it works with the C^0 uniformity of continuous mappings in compact-open topology. Even for spheres S^n of dimension $n > 1$ it does not work directly, but uses the iterated loop group construction of circles. Then their constructions were generalized for fibers over circles and spheres with parallel transport structures over **C**. Smooth Deligne cohomologies were studied on such groups [12].

Wrap groups of quaternion and octonion fibers as well as for wider classes of fibers over **R** or **C** were defined and various examples were given together with basic theorems in [21]. Studies of their structure were begun in [22]. This paper continues previous works of the author on this theme, where generalized loop groups of manifolds over **R**, **C** and **H** were investigated, but neither for fibers nor over octonions [23, 31, 29, 30].

Holomorphic functions of quaternion and octonion variables were investigated in [27, 28, 25]. There specific definition of super-differentiability was considered, because the quaternion skew field has the graded algebra structure. This definition of super-differentiability does not impose the condition of right or left super-linearity of a super-differential, since it leads to narrow class of functions. There are some articles on quaternion manifolds, but practically they undermine a complex manifold with additional quaternion structure of its tangent space (see, for example, [38, 51] and references therein). Quaternion manifolds were defined in another way in [25]. Applications of quaternions in mathematics and physics can be found in [9, 14, 15, 20].

Geometric loop groups have important applications in modern physical theories (see [17, 33] and references therein). Groups of loops are also intensively used in gauge theory. Wrap groups can be used in the membrane theory which is the generalization of the string (superstring) theory. Fiber bundles and sheaves and cohomologies over quaternions and octonions are interesting in such a respect, that they take into account spin and isospin structures on manifolds, because there is the embedding of the Lie group $U(2)$ into the quaternion skew field $\mathbf{H}$.

This article is devoted to constructions and investigations of cohomologies and sheaves of wrap groups. Moreover, over quaternions and octonions twisted cohomologies and sheaves are studied. Twisted analogs of bar resolutions of sheaves and smooth Deligne cohomology are investigated as well. This is done over twisted multiplicative groups. Previously the complex case and with loop groups of fiber bundles on spheres was only studied. This article treats the quaternion and octonion cases and wrap groups of general fiber bundles.

All main results of this paper are obtained for the first time and they are given in Theorems 34, 36, 44, 48.1, 55, 58, 60, Propositions 6, 14, 15, 19, 26, 27, 29, 32, Corollaries 7, 8, 33, 45 and 47. Here the notations and definitions and results from previous works [12, 13, 21, 22, 23, 31, 29, 30] are used.

2 Cohomologies of wrap groups

1. Remarks and Definitions. Consider a triangulated compact polyhedron M may be embedded into $\mathcal{A}_r^n$ and its sub-polyhedron S_M of codimension not less than two, $codim(S_M) \ge 2$, where $M \setminus S_M$ is a C^∞ smooth manifold such that $M \setminus S_M$ is dense in M. If the covering dimension (see Chapter 7 [10]) of $M \setminus S_M$ is $dim(M \setminus S_M) = b$, then by the definition M is of dimension b. Then S_M is called the singularity of M. A pseudo-manifold M is oriented, if $M \setminus S_M$ is oriented (see also §1.3.1 [21]).

If $M \setminus S_M$ is without boundary, then the triangulated pseudo-manifold M is called a pseudo-manifold cycle. If $(Y, \partial Y)$ is the pair consisting of a triangulated pseudo-manifold Y and a boundary ∂Y, such that $Y \setminus S_Y$ is a manifold with boundary $\partial Y \setminus S_Y$, ∂Y is a pseudo-manifold cycle with singularity $S_Y \cap \partial Y$, then $(Y, \partial Y)$ is called the triangulated pseudo-manifold with boundary.

A pre-sheaf F on a topological space X is a contra-variant functor F from the category of open subsets in X and their inclusions into a category of groups or rings (all either alternative or associative) such that $F(U)$ is a group or a ring for each U open in X and for each $U \subset V$ with U and V open in X there exists a homomorphism $s_{U,V} : F(V) \to F(U)$ such that $s_{U,U} = 1$ and $s_{U,V} s_{V,Y} = s_{U,Y}$ for each $U \subset V \subset Y$ with U, V, Y open in X.

Let F_x denotes the family of all elements $f \in F(U)$ for all U open in X with $x \in U$. Elements $f \in F(U)$ and $g \in F(V)$ are called equivalent if there exists an open neighborhood Y of x such that $s_{Y,U}(f) = s_{Y,V}(g)$. This generates an equivalence relation and a class of all equivalent elements with f is called a germ f_x of f at x. A set $\mathcal{F}_x$ of all germs of the pre-sheaf F at a point $x \in X$ is the inductive limit $\mathcal{F}_x = ind - \lim F(U)$ taken by all open neighborhoods U of x in X.

In the set $\mathcal{F}$ of all germs $\mathcal{F}_x$ take a base of topology consisting of all sets $\{f_x \in \mathcal{F}_x : x \in U\}$, where $f \in F(U)$. This induces a sheaf $\mathcal{S}$ generated by a pre-sheaf F.

A sheaf of groups or rings (all either alternative or associative) on X is a pair $(\mathcal{S}, h)$ satisfying Conditions $(S1 - S4)$:

$(S1)$ $\mathcal{S}$ is a topological space;

$(S2)$ $h : \mathcal{S} \to X$ is a local homeomorphism;

$(S3)$ for each $x \in X$ the set $\mathcal{F}_x = h^{-1}(x)$ is a group called a fiber of the sheaf $\mathcal{S}$ at a point x;

$(S4)$ the group or the ring operations are continuous, that is, $\mathcal{S}\Delta\mathcal{S} \ni (a, b) \mapsto ab^{-1} \in \mathcal{S}$ or $\mathcal{S}\Delta\mathcal{S} \ni (a, b) \mapsto ab \in \mathcal{S}$ and $\mathcal{S}\Delta\mathcal{S} \ni (a, b) \mapsto a + b \in \mathcal{S}$ are continuous respectively, where $\mathcal{S}\Delta\mathcal{S} := \{(a, b) : a, b \in \mathcal{S}, h(a) = h(b)\}$.

We can consider pre-sheafs and sheafs of different classes of smoothness, for example, H^t or H^t_p, when the corresponding defining sheaf and pre-sheaf mappings $s_{U,V}$, h and group operations are such and $\mathcal{S}$ and F are H^t or H^t_p differentiable spaces respectively (see also §1.3.2 [21]).

Consider a sheaf $\mathcal{S}_{N,G}$ generated by a pre-sheaf $U \mapsto \{f \in Hom^t_p((W^M E)_{t,H}, G) : supp(f) \subset U\}$, where U is open in N and $supp(f) \subset U$ means that there exists $y \in N$ and $\hat{\eta} \in H^t_p(\hat{M}, N)$ with $\hat{\eta}(\hat{s}_{0,q}) = x$ for each $q = 1, ..., k$ and $\hat{\eta}(\hat{s}_{0,q}) = y$ for each $q = k+1, ..., 2k$ and $\hat{\gamma} \in H^t_p(\hat{M}, \{\hat{s}_{0,q} : q = 1, ..., 2k\}; N, y)$ such that $\hat{\gamma} = \gamma \circ \Xi$ and $f = < \hat{\eta} \vee \hat{\gamma} >_{t,H}$, where the wrap group $(W^M E)_{t,H}$

is taken for a marked point $y \in N$, $\Xi : \hat{M} \to M$ is the quotient mapping as in [21].

In particular, we can take $G = \mathcal{A}_r^*$, and call $\mathcal{S}_{N,\mathcal{A}_r^*}$ the sheaf of infinitesimal holonomies, where $1 \le r \le 3$.

In view of Property $(P4)$ [21] for each non-singular points $y \in N$ and $u \in E_y$ in the fiber E_y of E over y there exists an $\mathcal{A}_r$ vector subspace H_u of the tangent bundle T_uE at u called a horizontal subspace of T_uE such that $\pi_*|_{H_u} : H_u \to T_yN$ is an isomorphism, where $\pi(u) = y$, $t' \ge [dim(E)/2] + 2$ or $t' = \infty$, since there exist generalized derivatives in the Sobolev space $H^{t'}$ (see §III.3 [34]). This is the case for all $y \in N$ and $u \in E_y$ when N and E are of class $H^{t'}$ instead of $H^{t'}_p$.

Due to $(P1)$ the family $\{H_u\}$ of horizontal subspaces of TE depends smoothly on u. Suppose that Y is a vector field in TE corresponding to a vector field X in TN such that $\pi_*(Y) = X$, then

$(CD1)$ $T_uE = H_u \oplus V_u$, where $V_u = \pi_*^{-1}(0) \subset T_uE$ is the space of vectors tangent to E_u at u. In accordance with $(P3)$ the horizontal spaces are G-equivariant, that is,

$(CD2)$ $H_{uz} = (R_z)_*H_u$, where R_z is the diffeomorphism of E given by the multiplication on z from the right and $(R_z)_*$ corresponds to the tangent mapping TR_z for the tangent fiber bundle TE.

A family $H = \{H_u \subset T_uE : u \in E, \pi(u) = y \in N\}$ is called the connection distribution of the principal fiber bundle $E(N, G, \pi, \Psi)$, if H_u depends smoothly on u and the Conditions $(CD1, CD2)$ are satisfied.

2. Definitions and Notes. Two smooth principal G fiber bundles E and E' with connection distributions (E, H) and (E', H') are called isomorphic if there exists an isomorphism $f : E \to E'$ of smooth principal G fiber bundles $f : E \to E'$ such that $f_*(H) = H'$.

A connection distribution H on E determines a parallel transport structure $\mathbf{P}^H$ on E posing $\mathbf{P}^H_{\hat{\gamma},u} \in H^t_p(\hat{M}, E)$ with $\mathbf{P}^H_{\hat{\gamma},u}(\hat{s}_{0,q}) = u$ for each $q = 1, ..., k$ and $\pi \circ \mathbf{P}^H_{\hat{\gamma},u} = \hat{\gamma}$ such that $T_x\mathbf{P}^H_{\hat{\gamma},u} =: (\mathbf{P}^H_{\hat{\gamma},u}(x), D\mathbf{P}^H_{\hat{\gamma},u}(x))$ for each $x \in \hat{M}$, where $\hat{\gamma} \in H^t_p(\hat{M}, \{\hat{s}_{0,q} : q = 1, ..., 2k\}; N, y_0)$, $D\mathbf{P}^H_{\hat{\gamma},u}(x) \in H_{\hat{\gamma}(x)}$, $T\mathbf{P}$ is the tangent mapping of $\mathbf{P}$ (see [19]).

Thus there exists a bijective correspondence between parallel transport structures and connection distributions on E. Therefore, the mapping $H \mapsto \mathbf{P}^H$ induces a bijective correspondence between isomorphism classes of parallel transport structures and connection distributions.

Using the exponential function on $\mathcal{A}_r$ gives $\exp(\mathcal{A}_r) = \mathcal{A}_r^*$ for $1 \le r \le 3$ (see §3 [27, 28]).

If E is a principal $\mathcal{A}_r^*$ fiber bundle with $1 \le r \le 3$, then for each $v \in V_u$ there exists a unique $z(v) \in \mathcal{A}_r$ such that $v = [d(y\ \exp(b\ z(v))/db]|_{b=0}$, where

$b \in \mathbf{R}$. Therefore, for each connection distribution $\{H_u : u \in E\}$ on E a differential 1-form w over $\mathcal{A}_r$ exists such that $w(X_h + X_v) = z(X_v)$ for each $X = X_h + X_v \in H_u \oplus V_u = T_u E$ and w is G-equivariant: $(R_z)^* w = w$ due to the G-equivariance of $\{H_u : u \in E\}$, here $G = \mathcal{A}_r^*$.

A differential 1-form w on E so that it is G-eququvariant and $w(X_v) = z(X_v)$ for each $X_v \in V_u$ is called a connection 1-form.

Two smooth principal G fiber bundles with connections (E, w) and (E', w') are called isomorphic, if there exists an isomorphism $f : E \to E'$ of smooth principal G fiber bundles such that $f^*(w') = w$.

For w there exists a connection distribution H^w on E for which $H^w_u = ker(w_u) \subset T_u E$, that induces a bijective correspondence between differential 1-forms and connection distributions on E. Hence $w \mapsto H^w$ produces a bijective correspondence between isomorphism classes of connections and connection distributions.

Then there exists a wrap group $(W^M E; N, \mathcal{A}_r^*, \nabla)_{t,H}$, where a parallel transport structure $\mathbf{P}$ is associated with the covariant differentiation ∇ of the connection w.

The curvature 2-form Ω over $\mathcal{A}_r$, $1 \le r \le 3$, of a connection 1-form w on a smooth principal fiber bundle $E(N, G, \pi, \Psi)$ over $\mathcal{A}_r$ is given by $\Omega(X, Y) = dw(hX, hY)$, where hX and hY are horizontal components of the vectors X and Y.

3. Remark. If $\eta \in H^t_p(K, E)$, $t \ge 1$, and ν is a differential form on E, then there exists its pull-back $\eta^* \nu$ which is a differential form on K, where K is an H^t_p-pseudo-manifold. For orientable K and E and an H^t_p diffeomorphism η of K onto E and ν with compact support $\int_K \eta^* \nu = \epsilon \int_E \nu$, where $\epsilon = 1$ if η preserves an orientation, $\epsilon = -1$ if η changes an orientation (see [5, 52, 25]). In particular, $K = E(M, G, \pi_M, \Psi^M)$ can be considered, $\eta = (\eta_0, \eta_1)$, $\eta_0 : M \to N$, $\eta : E(M) \to E(N)$, $\pi_N \circ \eta = \eta_0 \circ \pi_M$, $\eta_1 \circ pr_2 = pr_2 \circ \eta$, pr_2 is a projection in charts of E from E into G, $\eta_1 = id$ may be as well.

Suppose that M and E are an $\mathcal{A}_r$ holomorphic manifold and principal fiber bundle, such that E is orientable and $2^r - 1$-connected, which is not very restrictive due to Propositions 13 and Note 14 [22]. If $\hat{\gamma} \in H^t_p(\hat{M}, \{\hat{s}_{0,q} : q = 1, ..., k\}; N, y_0)$, then consider a path l_q joining the point $\hat{s}_{0,q}$ with $\hat{s}_{0,q+k}$, where $1 \le q \le k$, $\hat{l}_q : [0,1] \to \hat{M}$. Therefore, $\hat{p}_q := \hat{\gamma} \circ \hat{l}_q : [0,1] \to N$ and $\hat{p}^*_q w := (\hat{p}_q, id)^* w$ is a differential form on $[0,1]$, where w is an $\mathcal{A}_r$ holomorphic connection one-form on E. We get that $\hat{\gamma}^* w$ is a differential one-form on $\hat{M}$ and there exists its restriction $\nu_{\gamma,q} := \gamma^* w|_{\hat{l}_q[0,1]}$.

Then we have also $\gamma \in H^t_p(M, \{s_{0,q} : q = 1, ..., k\}; N, y_0)$ and $l_q : S^1 \to M$ and $p_q : S^1 \to N$ respectively, where $\hat{\gamma} = \gamma \circ \Xi$ (see [21]), S^1 is the unit circle in $\mathbf{C}$ with the center at zero, while $\mathbf{C} = \mathbf{C}_M$ is embedded into $\mathcal{A}_r$ as $\mathbf{R} \oplus M\mathbf{R}$

with $M \in \mathcal{A}_r$, $Re(M) = 0$, $|M| = 1$, when $2 \le r \le 3$.

Since w is $\mathcal{A}_r$ holomorphic, then $\int_\phi w$ does not depend on a rectifiable curve ϕ but only on the initial and final points $\phi(0)$ and $\phi(1)$, $\phi : [0,1] \to E$ (see Theorems 2.15 and 3.10 in [27, 28] and [30]).

Consider now the principal fiber bundle E with the structure group $\mathcal{A}_r^*$, where $1 \le r \le 3$. Then the pull-back p_q^*E of the bundle E is a trivial $\mathcal{A}_r^*$-bundle over S^1. The latter bundle carries a pull-back connection differential one-form p_q^*w. Take the pull-back $\rho^*(p_q^*w)$ one form, where $\rho : S^1 \to p_q^*E$ is a trivialization of the fiber bundle $p_q^*E \to S_1$.

The parallel transport structure $\mathbf{P}_{\hat{\gamma},u}(x)$ for (M,E) with $x \in \hat{M}$ induces the parallel transport structures $\mathbf{P}_{\hat{p}_q,u^*}(s)$ for (S^1, p_q^*E) with $s \in [0,1]$ for each $q = 1, ..., k$, where $p_q(u^*) = u$. Then the holonomy along γ is given by

(H) $h(\gamma) = (h_1, ..., h_k) \in G^k$ with $h_q = h_q(\gamma) = \exp[-\int_{S^1} \rho^*(p_q^*w)]$ for each $q = 1, ..., k$.

If $\zeta : S^1 \to p_q^*E$ is another trivialization and $f : S^1 \to \mathbf{C}_M^*$ satisfies $\zeta = f\rho$, so that $f(v) = \exp(M2\pi\theta(v))$, where $\theta(v) \in \mathbf{R}$, $M2\pi d\theta(v) = dLn(f(v))$, $v \in S^1$, $\int_{S^1} d\theta$ is an integer number, since $\mathbf{R}$ is the center of the algebra $\mathcal{A}_r$, where Ln is the natural logarithmic function over $\mathcal{A}_r$ (see §3.7 and Theorem 3.8.3 [28] and [27, 32]). Therefore, Formula (H) is independent of a trivialization ρ, since $\zeta^*(p_q^*w) = \rho^*(p_q^*w) + dLn(f)$, but $\exp[\int_{S^1} dLn(f)] = 1$.

4. Non-associative bar construction. Let G be a topological group not necessarily associative, but alternative:

$(A1)$ $g(gf) = (gg)f$ and $(fg)g = f(gg)$ and $g^{-1}(gf) = f$ and $(fg)g^{-1} = f$ for each $f, g \in G$

and having a conjugation operation which is a continuous automorphism of G such that

$(C1)$ $conj(gf) = conj(f)conj(g)$ for each $g, f \in G$,

$(C2)$ $conj(e) = e$ for the unit element e in G.

If G is of definite class of smoothness, for example, H_p^t differentiable, then $conj$ is supposed to be of the same class. For commutative group in particular it can be taken the identity mapping as the conjugation. For $G = \mathcal{A}_r^*$ it can be taken $conj(z) = \tilde{z}$ the usual conjugation for each $z \in \mathcal{A}_r^*$, where $1 \le r \le 3$.

Denote by $\Delta^n := \{(x_0, ..., x_n) \in \mathbf{R}^{n+1} : x_j \ge 0, x_0 + x_1 + ... + x_n = 1\}$ the standard simplex in $\mathbf{R}^{n+1}$. Consider $(AG)_n$ as the quotient of the disjoint union $\bigcup_{k=0}^n (\Delta^k \times G^{k+1})$ by the equivalence relations

(1) $(x_0, ..., x_k, g_0, ..., g_k) \sim (x_0, ..., x_j + x_{j+1}, ..., x_k, g_0, ..., \hat{g}_j, ..., g_k)$ for $g_j = g_{j+1}$ or $x_j = 0$ with $0 \le j < k$; $(x_0, ..., x_k, g_0, ..., g_k) \sim (x_0, ..., x_{k-1} + x_k, g_0, ..., g_{k-1})$ for $g_{k-1} = g_k$ or $x_k = 0$.

Consider non-homogeneous coordinates $0 \le t_1 \le t_2 \le ... \le t_k \le 1$ on the simplex Δ^k related with the barycentric coordinates by the formula

$t_j = x_0 + x_1 + ... + x_{j-1}$ and $h_0 := g_0$, $h_j = g_{j-1}^{-1} g_j$ for $j > 0$ on G^{k+1}. Hence $h_0 h_1 = g_0(g_0^{-1} g_1) = g_1$, $(h_0 h_1) h_2 = g_1(g_1^{-1} g_2) = g_2$ and by induction $((...(h_0 h_1)...)h_{k-1})h_k = g_{k-1}(g_{k-1}^{-1} g_k) = g_k$.

Then equivalence relations (1) take the form:

(2) $(t_1, ..., t_k, h_0[h_1|...|h_k]) \sim (t_2, ..., t_k, (h_0 h_1)[h_2|...|h_k])$ for $t_1 = 0$ or $h_0 = e$;

$(t_1, ..., t_k, h_0[h_1|...|h_k]) \sim (t_1, ..., \hat{t}_j, ..., t_k, h_0[h_1|...|h_j h_{j+1}|...|h_k])$ for $t_j = t_{j+1}$ or $h_j = e$;

$(t_1, ..., t_k, h_0[h_1|...|h_k]) \sim (t_1, ..., t_{k-1}, h_0[h_1|...|h_{k-1}])$ for $t_k = 1$ or $h_k = e$.

Denote by $|x_0, ..., x_k, g_0, ..., g_k|$ the equivalence class of the sequence $(x_0, ..., x_k, g_0, ..., g_k)$; by $|t_1, ..., t_k, h_0[h_1|...|h_k]|$ denote the equivalence class of the sequence $(t_1, ..., t_k, h_0[h_1|...|h_k])$.

Then the space AG is the quotient of $\bigcup_{k=0}^{\infty} \Delta^k \times G^{k+1}$ by the above equivalence relations (1), where $(\Delta^k \times G^{k+1}) \cap (\Delta^m \times G^{m+1})$ is empty for $k \ne m$.

Introduce in G^{n+1} the equivalence relation $\mathcal{Y}$:

(3) $(g_0, ..., g_n) \mathcal{Y} (q_0, ..., q_n)$ if and only if there exist $p_1, ..., p_k \in G$ with $k \in \mathbf{N}$ such that $g_j = p_k(p_{k-1}...(p_2(p_1 q_j))....)$ for each $j = 0, ..., n$.

Evidently this relation is reflexive: $(g_0, ..., g_n) \mathcal{Y} (g_0, ..., g_n)$ with $p_1 = e$ and $k = 1$. It is symmetric due to the alternativity of G, since $g_j = p_k(p_{k-1}...(p_2(p_1 q_j))....)$ is equivalent with $q_j = p_1^{-1}(p_2^{-1}...(p_{k-1}^{-1}(p_k^{-1} g_j))...)$ for each $j = 0, ..., n$. This relation is transitive: $(g_0, ..., g_n) \mathcal{Y} (q_0, ..., q_n)$ and $(q_0, ..., q_n) \mathcal{Y} (f_0, ..., f_n)$ implies $(g_0, ..., g_n) \mathcal{Y} (f_0, ..., f_n)$, since from

$g_j = p_k(p_{k-1}...(p_2(p_1 q_j))....)$ and $q_j = s_l(s_{l-1}...(s_2(s_1 f_j))....)$

it follows $g_j = p_k(p_{k-1}...(p_2(p_1(s_l(s_{l-1}...(s_2(s_1 f_j))....)))....)$ for each $j = 0, ..., n$, where $k, l \in \mathbf{N}$, $p_1, ..., p_k, s_1, ..., s_l \in G$. In a particular case of an associative group G parameters $k = 1$ and $l = 1$ can be taken.

Consider in $\bigcup_{k=0}^{n} \Delta^k \times G^k$ the equivalence relations:

(4) $(x_0, ..., x_k, [g_0 : ... : g_k]) \sim (x_0, ..., x_j + x_{j+1}, ..., x_k, [g_0 : ... : \hat{g}_j : ... : g_k])$ for $g_j = g_{j+1}$ or $x_j = 0$ with $0 \le j < k$; $(x_0, ..., x_k, g_0, ..., g_k) \sim (x_0, ..., x_{k-1} + x_k, [g_0 : ... : g_{k-1}])$ for $g_{k-1} = g_k$ or $x_k = 0$, where $[g_0 : ... : g_k] := \{(q_0, ..., q_k) \in G^{k+1} : (q_0, ..., q_k) \mathcal{Y} (g_0, ..., g_k)\}$ denotes the equivalence class of $(g_0, ..., g_k)$ by the equivalence relation $\mathcal{Y}$. Put $(BG)_n$ to be the quotient of $\bigcup_{k=0}^{n} \Delta^k \times G^k$ by equivalence relations (4).

Using the inhomogeneous coordinates on $(BG)_n$ rewrite the equivalence relation (4) in the form:

(5) $(t_1, ..., t_k, [h_1|...|h_k]) \sim (t_2, ..., t_k, [h_2|...|h_k])$ for $t_1 = 0$ or $h_0 = e$;

$(t_1, ..., t_k, h_0[h_1|...|h_k]) \sim (t_1, ..., \hat{t}_j, ..., t_k, [h_1|...|h_j h_{j+1}|...|h_k])$ for $t_j = t_{j+1}$ or $h_j = e$;

$(t_1, ..., t_k, [h_1|...|h_k]) \sim (t_1, ..., t_{k-1}, [h_1|...|h_{k-1}])$ for $t_k = 1$ or $h_k = e$.

Denote by $|x_0, ..., x_k, [g_0 : ... : g_k]|$ the equivalence class of the sequence $(x_0, ..., x_k, [g_0 : ... : g_k])$; by $|t_1, ..., t_k, [h_1|...|h_k]|$ denote the equivalence class

of the sequence $(t_1, ..., t_k, [h_1|...|h_k])$. Then BG is the quotient of the disjoint union $\bigcup_{k=0}^{\infty} \Delta^k \times G^k$ by the equivalence relations (4).

Then there exists the projection $\pi_B^A : AG \to BG$ by the formula:

(6) $\pi_B^A : |x_0, ..., x_k, g_0, ..., g_k| \mapsto |x_0, ..., x_k, [g_0 : ... : g_k]|$ or in the non-homogeneous coordinates by $\pi_B^A : |t_1, ..., t_k, h_0[h_1|...|h_k]| \mapsto |t_1, ..., t_k, [h_1|...|h_k]|$.

The conjugation in G induces that of in AG and BG such that:

$conj(t_1, ..., t_k, h_0[h_1|...|h_k]) := (t_1, ..., t_k, conj(h_0)[conj(h_1)|...|conj(h_k)])$ and

$conj(t_1, ..., t_k, [h_1|...|h_k]) := (t_1, ..., t_k, [conj(h_1)|...|conj(h_k)])$.

Suppose that

$(A2)$ $\hat{G} = \hat{G}_0 i_0 \oplus \hat{G}_1 i_1 \oplus ... \oplus \hat{G}_{2^r-1} i_{2^r-1}$ such that G is a multiplicative group of a ring $\hat{G}$ with the multiplicative group structure, where $G_j = \hat{G}_j \setminus \{0\}$, $\hat{G}_0, ..., \hat{G}_{2^r-1}$ are pairwise isomorphic commutative associative rings and $\{i_0, ..., i_{2^r-1}\}$ are generators of the Cayley-Dickson algebra $\mathcal{A}_r$, $1 \le r \le 3$ and $(y_l i_l)(y_s i_s) = (y_l y_s)(i_l i_s)$ is the natural multiplication of any pure states in $\hat{G}$ for $y_l \in \hat{G}_l$. If a group G and a ring $\hat{G}$ satisfy Conditions $(A1, A2, C1, C2)$, then we call it a twisted group and a twisted ring over the set of generators $\{i_0, ..., i_{2^r-1}\}$, where $1 \le r \le 3$. The unit element of G denote by e. For example, $G = (\mathcal{A}_r^*)^n$ and $\hat{G} = \mathcal{A}_r^n$.

5. Definitions. Let $N_.$ be a family $\{N_n : n \in \mathbf{N}\}$ of either C^∞ smooth or $H_p^{t'}$ manifolds together with either C^∞ or $H_p^{t'}$ mappings $\partial_j : N_n \to N_{n-1}$ and $s_j : N_n \to N_{n+1}$ for each $j = 0, 1, ..., n$ satisfying the identities:

(1) $\partial_k \partial_j = \partial_{j-1} \partial_k$ for each $k < j$,

(2) $s_k s_j = s_{j+1} s_k$ for each $k \le j$,

(3) $\partial_k s_j = s_{j-1} \partial_k$ for $k < j$, $\partial_k s_j = id|_{N_n}$ for $k = j, j+1$, $\partial_k s_j = s_j \partial_{k-1}$ for $k > j+1$, then $N_.$ is called a simplicial either C^∞ smooth or $H_p^{t'}$ manifold.

The geometric realization $|N_.|$ of $N_.$ consists of $\coprod_{n \ge 0} \Delta^n \times N_n / \mathcal{E}$, where $\mathcal{E}$ is the equivalence relation generated by $(\partial^j x, y) \mathcal{E} (x, \partial_j y)$ for $(x, y) \in \Delta^{n-1} \times N_n$, $(s^j x, y) \mathcal{E} (x, s_j y)$ for $(x, y) \in \Delta^{n+1} \times N_n$, where $\coprod$ denotes the disjoint union of sets, the maps $\partial^j : \Delta^{n-1} \to \Delta^n$ and $s^j : \Delta^{n+1} \to \Delta^n$ are such that $\partial^j(x_0, ..., x_{n-1}) = (x_0, ..., x_{j-1}, 0, x_j, ..., x_{n-1})$ and $s^j(x_0, ..., x_{n+1}) = (x_0, ..., x_{j-1}, x_j + x_{j+1}, x_{j+2}, ..., x_{n+1})$ in barycentric coordinates.

A C^∞ or $H_p^{t'}$ space structure on the geometric realization $|N_.|$ of $N_.$ consists of all continuous C^∞ $\mathbf{R}$-valued or $H_p^{t'}$ $\mathcal{A}_r$ valued functions f on $|N_.|$ respectively, that is the composition $\coprod_{n \ge 0} (\Delta^n - \partial \Delta^n) \times N_n \hookrightarrow \coprod_{n \ge 0} \Delta^n \times N_n \xrightarrow{q} |N_.| \xrightarrow{f} \mathcal{A}_r$ is either C^∞ or $H_p^{t'}$, where q denotes the quotient mapping, $r = 0$ or $1 \le r \le 3$ correspondingly, $\mathcal{A}_0 = \mathbf{R}$, $\mathcal{A}_1 = \mathbf{C}$, $\mathcal{A}_2 = \mathbf{H}$, $\mathcal{A}_3 = \mathbf{O}$.

6. Proposition. *If a group G satisfies Conditions 4$(A1, A2, C1, C2)$, then sets AG and BG can be supplied with group structures and they are twisted for $2 \le r \le 3$. If G is a topological Hausdorff or H_p^t differentiable*

alternative for $r=3$ or associative for $0 \le r \le 2$ group, then AG and BG are topological Hausdorff or C^∞ or H^t_p differentiable alternative for $r=3$ or associative for $0 \le r \le 2$ groups respectively.

Proof. Define on AG and BG group structures. Introduce a homeomorphism pairing: $\Delta^n \times \Delta^k \to \Delta^{n+k}$, where σ is a permutation of the set $\{1,2,...,n+m+1\}$ such that $t_{\sigma(1)} \le t_{\sigma(2)} \le ... \le t_{\sigma(n+k+1)}$, $\sigma \in S_{n+k+1}$, S_m denotes the symmetric group of all permutations of the set $\{1,...,m\}$. Define the multiplication for pure states in AG:

(1) $|t_1,...,t_n,h_0[h_1|...|h_n]| * |t_{n+1},...,t_{n+k+1},h_{n+k+2}[h_{n+1}|...|h_{n+k+1}]| :=$
$|t_{\sigma(1)},...,t_{\sigma(n+k+1)},(-1)^{q(\sigma)}(h_0h_{n+k+2})[h_{\sigma(1)}|...|h_{\sigma(n+k+1)}]|$,
where $h_l = y_l i_{j(l)}$, $y_l \in G_{j(l)}$ for each $l=0,...,2^r-1$, $q(\sigma) \in \mathbf{Z}$ is such that
$(-1)^{q(\sigma)} i_{j(0)}(i_{j(1)}...(i_{j(n+k+1)}i_{j(n+k+2)})...) = (i_{j(\sigma(0))}i_{j(\sigma(n+k+2))})(i_{j(\sigma(1))}\cdots$
$(i_{j(\sigma(n+k)))}i_{j(\sigma(n+k+1))})...)$ in $\mathcal{A}_r$; while in BG:

(2) $|t_1,...,t_n,[h_1|...|h_n]| * |t_{n+1},...,t_{n+k+1},[h_{n+1}|...|h_{n+k+1}]| :=$
$|t_{\sigma(1)},...,t_{\sigma(n+k+1)},(-1)^{p(\sigma)}[h_{\sigma(1)}|...|h_{\sigma(n+k+1)}]|$,
where $h_l = y_l i_{j(l)}$, $y_l \in G_{j(l)}$, $p(\sigma) \in \mathbf{Z}$ is such that
$(-1)^{p(\sigma)} i_{j(1)}(i_{j(2)}...(i_{j(n+k)}i_{j(n+k+1)})...) =$
$i_{j(\sigma(1))}(i_{j(\sigma(2))}...(i_{j(\sigma(n+k))}i_{j(\sigma(n+k+1))})...)$ in $\mathcal{A}_r$.

Define also an addition in $A\hat{G}$:

(1′) $|t_1,...,t_n,h_0[h_1|...|h_n]| + |t_{n+1},...,t_{n+k+1},h_{n+k+2}[h_{n+1}|...|h_{n+k+1}]|$
$:= |t_{\sigma(1)},...,t_{\sigma(n+k+1)},(-1)^{q(\sigma)}(h_0+h_{n+k+2})[h_{\sigma(1)}|...|h_{\sigma(n+k+1)}]|$
with $j(0)=j(n+k+2)$; as well as an addition in $B\hat{G}$:

(2′) $|t_1,...,t_n,[h_1|...|h_n]| + |t_{n+1},...,t_{n+k+1},[h_{n+1},...,h_{n+k+1}]|$
$= |t_{\sigma(1)},...,t_{\sigma(n+k+1)},(-1)^{p(\sigma)}[h_{\sigma(1)}|...|h_{\sigma(n+k+1)}]|$
for pure states $h_l = y_l i_{j(l)}$, $y_l \in \hat{G}_l$ for each $l=0,...,2^r-1$. The multiplications $(1,2)$ extend to that of rings in the natural way, when some pure states are zero, hence due to the distributivity on the entire ring as well.

Since $\hat{G}$ is the ring, then these multiplications have unique extensions on AG and BG. Verify, that AG and BG become groups with multiplications (1) and (2) respectively.

Due to $(1,2)$ we get

(3) $v * conj(v) = |t_1,...,t_k,(h_0conj(h_0))[(h_1conj(h_1))|...|(h_kconj(h_k))]|$ for each $v = |t_1,...,t_k,h_0[h_1|....|h_k]|$ in AG, while

$w * conj(w) = |t_1,...,t_k,[(h_1conj(h_1))|...|(h_kconj(h_k))]|$

for each $w = |t_1,...,t_k,[h_1|....|h_k]|$ in BG, where $hconj(h) \in \hat{G}_0$ for each $h \in G$, but $\hat{G}_0$ is the center of the ring $\hat{G}$: $ab=ba$ for each $a \in \hat{G}_0$ and $b \in \hat{G}$. The Moufang identities in $\mathcal{A}_r$ for $r=3$ (see [16]) induces that of in G such that

(4) $(xyx)z = x(y(xz))$ and $(x^{-1}yx)z = x^{-1}(y(xz))$;

(5) $z(xyx) = ((zx)y)x$ and $z(x^{-1}yx) = ((zx^{-1})y)x$;

(6) $(xy)(zx)=x(yz)x$ and $(x^{-1}y)(zx)=x^{-1}(zy)x$, since

(7) $x^{-1}=conj(x)(x\ conj(x))^{-1}$,

where $(xconj(x))\in G_0$.

The unit element in AG is

$\mathbf{e}:=\{|t_1,...,t_k,e[e|...|e]|\in (AG)_k : k=0,1,...\}$,

where $i_0=1$, since

$|t_1,...,t_n,h_0[h_1|...|h_n]|*|t_{n+1},...,t_{n+k+1},e[e|...|e]|=$
$|t_{n+1},...,t_{n+k+1},e[e|...|e]|*|t_1,...,t_n,h_0[h_1|...|h_n]|=$
$|t_1,...,t_n,h_0[h_1|...|h_n]|$ due to equivalence relations 4(2), $(1,...,1,e[e|...|e])\in |t_1,...,t_k,e[e|...|e]|$.

The ring $\hat{G}$ is $\mathbf{Z_2}$ graded in the sense that elements $y_lj_l\in \hat{G}_lj_l$ are even for $l=0$ and odd for $l=1,...,2^r-1$: $(y_0i_0)(y_ly_l)=(y_li_l)(y_0i_0)=(y_0y_l)i_l\in \hat{G}_li_l$ for each $0\le l\le 2^r-1$, $(y_li_l)^2=-y_l^2i_0\in \hat{G}_0i_0$, $(y_li_l)(y_ki_k)=-(y_ki_k)(y_li_l)=(y_ly_k)i_s\in \hat{G}_si_s$ for $1\le l\ne k\le 2^r-1$, where $i_s=i_li_k$. For each pure states $g_0,...,g_k\in \hat{G}$ their product $(...(g_0g_1)g_2...)g_k$ is a pure state, consequently, sets AG and BG are $\mathbf{Z_2}$ graded analogously to $\hat{G}$ having even and odd elements such that

(8) $A\hat{G}=(A\hat{G}_0)i_0\oplus (A\hat{G}_1)i_1\oplus ...\oplus (A\hat{G}_{2^r-1})i_{2^r-1}$ and

(9) $B\hat{G}=(B\hat{G}_0)i_0\oplus (B\hat{G}_1)i_1\oplus ...\oplus (B\hat{G}_{2^r-1})i_{2^r-1}$. Each AG_j and BG_j is an associative topological Hausdorff or H^t_p differentiable group isomorphic with AG_0 or BG_0 correspondingly for each j, since G_j are commutative and associative (see also Appendix B4 in [13]), where G_0 denotes the multiplicative group of the ring $\hat{G}_0$. Therefore, AG and BG are the multiplicative groups of the rings $A\hat{G}$ and $B\hat{G}$.

If $a\in AG_0$ or $a\in BG_0$, then $ab=ba$ for each $b\in AG$ or BG respectively. From Definition 6 it follows, that they are C^∞ or H^t_p groups, when such is G.

The inverse element is

(10) $\{|t_1,...,t_k,h_0[h_1|...|h_k]|:k\}^{-1}=\{|t_1,...,t_k,h_0^{-1}[h_1^{-1}|...|h_k^{-1}]|:k\}$

due to $(2,6)$, since

$(h_0(h_1...(h_{k-1}h_k)...))((...(h_k^{-1}h_{k-1}^{-1})...h_1^{-1})h_0^{-1})=$
$(...(((h_0h_0^{-1})(h_1h_1^{-1}))(h_2h_2^{-1})...)(h_kh_k^{-1})=e$

for pure states for each k in view of the Moufang identities $(4-6)$. In general it follows from $(3,8,9)$, since $v*conj(v)\in AG_0$ or BG_0 for $v\in AG$ or $v\in BG$ respectively, hence $v^{-1}=conj(v)(v*conj(v))^{-1}=\{|t_1,...,t_k,h_0[h_1|...|h_k]|:k\}^{-1}$.

In view of $(8,9)$ and the existence of an inverse element we get, that AG is alternative, since $1\le r\le 3$. Putting $h_0=1$ and applying the equivalence relation $\mathcal{Y}$ we get, that BG is also an alternative group, since the multiplicative group $\{i_0,...,i_7\}$ is alternative. If G is associative, for example, when $1\le r\le 2$, then AG and BG are associative, since the multiplicative

group $\{i_0, i_1, i_2, i_3\}$ is associative.

Thus, groups AG and BG are $\mathbf{Z_2}$ graded, hence they are twisted over $\{i_0, ..., i_{2^r-1}\}$. Consider for $\hat{G}$ multiplication and addition operations, then they induce them for AG and BG as above. It follows, that $E\hat{G}$ and $B\hat{G}$ are twisted rings.

7. Corollary. *Let suppositions of Proposition 6 be satisfied, then AB^mG and B^mG are topological or C^∞ or H^t_p differentiable groups respectively for each $m \ge 1$. Moreover, all maps in the short exact sequence $e \to B^aG \to AB^aG \to B^{a+1}G \to e$ are continuous or C^∞ or H^t_p correspondingly.*

Proof. Define differentiable space structure by induction. Suppose that it is defined on B^aG and $\Delta^k \times (B^aG)^m$ for $k, m \ge 0$, where $a \ge 1$. Then $f : AB^aG \to \mathcal{A}_r$ is C^∞ or H^t_p if the composition $\coprod_{n\ge 0}(\Delta^n - \partial\Delta^n) \times (B^aG)^{n+1} \xrightarrow{q_A} AB^aG \xrightarrow{f} \mathcal{A}_r$ is either C^∞ or $H^{t'}_p$, while $f : B^{a+1}G \to \mathcal{A}_r$ is C^∞ or H^t_p if the composition $\coprod_{n\ge 0}(\Delta^n - \partial\Delta^n) \times (B^aG)^n \xrightarrow{q_B} B^{a+1}G \xrightarrow{f} \mathcal{A}_r$ is either C^∞ or $H^{t'}_p$, where $0 \le r \le 3$.

A function $f : \Delta^k \times (B^{a+1}G)^m \to \mathcal{A}_r$ is C^∞ or H^t_p if the composition $\Delta^k \times (\coprod_{n\ge 0}(\Delta^n - \partial\Delta^n) \times (B^aG)^n)^m \xrightarrow{id\times(q_B)^m} \Delta^k \times (B^{a+1}G)^m \xrightarrow{f} \mathcal{A}_r$ is either C^∞ or $H^{t'}_p$. From this it follows that all maps in the short exact sequences are of the same class of smoothness.

Then the mappings $B^aG \times B^aG \to B^aG$ and $AB^aG \times AB^aG \to AB^aG$ of the form $(f, g) \mapsto fg^{-1}$ are C^0 or C^∞ or H^t_p in respective cases due to Formulas $6(1-3, 8-10)$ (see also §1.3.2 [21] and §1 and Appendix B in [13]).

8. Corollary. *If a group G satisfies Conditions $4(A1, A2, C1, C2)$, then there exist H^t_p groups $AB^a(W^{M,\{s_{0,q}:q=1,...,k\}}E)_{t,H}$ and $B^a(W^{M,\{s_{0,q}:q=1,...,k\}}E)_{t,H}$ for each $a \in \mathbf{N}$.*

Proof. The wrap group $(W^{M,\{s_{0,q}:q=1,...,k\}}E; N, G, \mathbf{P})_{t,H}$ is a principal G^k bundle over $(W^{M,\{s_{0,q}:q=1,..,k\}}N)_{t,H}$, where $(W^{M,\{s_{0,q}:q=1,...,k\}}N)_{t,H}$ is commutative and associative (see Proposition 7(1) in [22]).

If $g \in \hat{G}$, then g has the decomposition $g = g_0 i_0 + ... + g_{2^r-1} i_{2^r-1}$ with $g_j \in \hat{G}_j$ for each $j = 0, 1, ..., 2^r - 1$ and

(1) $g_0 = (g + (2^r - 2)^{-1}\{-g + \sum_{s=1}^{2^r-1} i_s(g i_s^*)\})/2$ and

(2) $g_j = (i_j(2^r - 2)^{-1}\{-g + \sum_{s=1}^{2^r-1} i_s(g i_s^*)\} - g i_j)/2$

for each $j = 1, ..., 2^r - 1$. Therefore, each $g_0, ..., g_{2^r-1}$ has analytic expressions through g due to Formulas $(1, 2)$. Fix this representations. Then the H^t_p differentiable parallel transport structure $\mathbf{P}$ with the groups G induces the H^t_p differentiable parallel transport structures ${}_j\mathbf{P}$ with groups G_j.

Since $\hat{G}^k$ is isomorphic with $\bigoplus_{0\le j(1),...,j(k)\le 2^r-1}(\hat{G}_{j(1)} i_{j(1)}, ..., \hat{G}_{j(k)} i_{j(k)})$ which is isomorphic with $\bigoplus_{0\le j(1),...,j(k)\le 2^r-1} \hat{G}^k_0(i_{j(1)}, ..., i_{j(k)})$, then $(W^{M,\{s_{0,q}:q=1,...,k\}}E; N, G, \mathbf{P})_{t,H}$ is isomorphic with a group

$\{f=(f_1,...,f_k)\in\bigoplus_{0\le j(1),...,j(k)\le 2^r-1}[(W^{M,\{s_{0,q}:q=1,...,k\}}E;N,G_0,\mathbf{P})_{t,H}\cup\{0\}]$ $(i_{j(1)},...,i_{j(k)}) : f_1\ne 0,...,f_k\ne 0\}$, where $(i_{j(1)},...,i_{j(k)})\in(\mathcal{A}_r^*)^k$ and $(\mathcal{A}_r^*)^k$ has the embedding into the family of all $k\times k$ matrices with entries in $\mathcal{A}_r$ as diagonal matrices, $(W^{M,\{s_{0,q}:q=1,...,k\}}E;N,G_j,\mathbf{P})_{t,H}$ is commutative for each $j=0,...,2^r-1$ due to Theorem 6 [21].

The construction of Proposition 5 above has the natural generalization for G^k instead of G such that

$A\hat{G}^k=\bigoplus_{0\le j(1),...,j(k)\le 2^r-1}(A\hat{G}_{j(1)}i_{j(1)},...,A\hat{G}_{j(k)}i_{j(k)})$

which is isomorphic with $\bigoplus_{0\le j(1),...,j(k)\le 2^r-1}A\hat{G}_0^k(i_{j(1)},...,i_{j(k)})$, also $B\hat{G}^k$ is isomorphic with $\bigoplus_{0\le j(1),...,j(k)\le 2^r-1}B\hat{G}_0^k(i_{j(1)},...,i_{j(k)})$, consequently, $A(W^{M,\{s_{0,q}:q=1,...,k\}}E;N,G,\mathbf{P})_{t,H}$ is isomorphic with a group $\{v\in\bigoplus_{0\le j(1),...,j(k)\le 2^r-1}[A(W^{M,\{s_{0,q}:q=1,...,k\}}E;N,G_0,\mathbf{P})^k_{t,H}\cup\{0\}](i_{j(1)},...,i_{j(k)}) :$ $v_n=|t_1,...,t_n,h_0[h_1|...|h_n]|, h_j\ne 0\forall j,\forall n\}$ and $B(W^{M,\{s_{0,q}:q=1,...,k\}}E;N,G,\mathbf{P})_{t,H}$ is isomorphic with $\{v\in\bigoplus_{0\le j(1),...,j(k)\le 2^r-1}[B(W^{M,\{s_{0,q}:q=1,...,k\}}E;N,G_0,\mathbf{P})^k_{t,H}\cup\{0\}](i_{j(1)},...,i_{j(k)}) :$ $v_n=|t_1,...,t_n,[h_1|...|h_n]|, h_j\ne 0\forall j,\forall n\}$. Continuing this by induction on a and using Corollary 7 we get the statement of this corollary for each $a\in\mathbf{N}$.

9. Lemma. *Let N be a C^∞ or H^t_p manifold over $\mathcal{A}_r$ with $0\le r\le 3$ and G a C^∞ or H^t_p differentiable group. If $f:N\to BG$ is a mapping such that for each $y\in N$ there exists an open neighborhood V of y in N such that $f|_V=|f_0,f_1,...,f_n,[g_1|...|g_n]|$ with $f_0,...,f_n$ being C^∞ or H^t_p differentiable mappings, then f is either C^∞ or H^t_p differentiable mapping correspondingly.*

Proof. If $h:BG\to\mathcal{A}_r$ is a C^∞ or H^t_p mapping, then for each $n\ge 1$ the composition $\Delta^n\times G^n\xrightarrow{q_B}BG\xrightarrow{h}\mathcal{A}_r$ is of the corresponding class. For the commutative diagram consisting of $N\xrightarrow{f}BG\xrightarrow{h}\mathcal{A}_r$ and $N\xrightarrow{\bar{f}}\Delta^n\times G^n\xrightarrow{q_B}BG$ and $f=q_b\circ\bar{f}$, where $\bar{f}:=(f_0,...,f_n,h_1,...,h_n)$ both $\bar{f}$ and $h\circ q_B$ are continuous C^∞ or H^t_p. Then the composition $h\circ f=h\circ q_B\circ\bar{f}$ is continuous and either C^∞ or H^t_p, where as usually $h\circ f(y):=h(f(y))$. Thus $f:N\to BG$ is continuous either C^∞ or H^t_p respectively.

10. Twisted bar resolution and hypercohomologies. For a twisted group G satisfying Conditions $4(A1,A2,C1,C2)$ the composition of the short exact sequences

(1) $e\to B^aG\to AB^aG\to B^{a+1}G\to e$ induces the long exact sequence

(2) $e\to G\to AG\xrightarrow{\sigma}ABG\xrightarrow{\sigma}AB^2G\xrightarrow{\sigma}...\xrightarrow{\sigma}AB^aG\xrightarrow{\sigma}...$,

where for each $a\ge 0$ the homomoprhism $\sigma:AB^aG\to AB^{a+1}G$ is the composition $AB^aG\to B^{a+1}G\to AB^{a+1}G$ of the surjection $AB^aG\to B^{a+1}G$ and the monomorphism $B^{a+1}G\to AB^{a+1}G$.

In view of Corollary 7 the short exact sequence (2) is a C^∞ or H^t_p B^aG-extension of $B^{a+1}G$. Hence the long exact sequence (2) induces the long exact sequence of twisted sheaves

(3) $e \to G_N \to AG_N \xrightarrow{\sigma} ABG_N \xrightarrow{\sigma} AB^2G_N \xrightarrow{\sigma} ... \xrightarrow{\sigma} AB^aG_N \xrightarrow{\sigma}$ ..., which we will call the (twisted) bar resolution of the sheaf G_N, where G_N denotes the sheaf of C^∞ or H^t_p functions on N with values in G.

Suppose that $\mathcal{S}^*$ and $\mathcal{F}^*$ are complexes of sheaves of $\mathcal{G}$-modules, where $\mathcal{G}$ is a sheaf of rings, where $\mathcal{S}^*$ and $\mathcal{F}^*$ and $\mathcal{G}$ may be simultaneously twisted over $\{i_0, ..., i^{2^r-1}\}$. Then a homomorphism mapping $\sigma : \mathcal{S}^* \to \mathcal{F}^*$ of such complexes induces a mapping of cohomology sheaves $\mathsf{H}^j(\sigma) : \mathsf{H}^j(\mathcal{S}^*) \to \mathsf{H}^j(\mathcal{F}^*)$, where $\mathsf{H}^j(\mathcal{S}^*)$ is the sheaf associated with the pre-sheaf $U \mapsto [ker(\Gamma(U, \mathcal{F}^j)) \to \Gamma(U, \mathcal{F}^{j+1}))]/im[(\Gamma(U, \mathcal{F}^{j-1})) \to \Gamma(U, \mathcal{F}^j))]$, where $\Gamma(U, \mathcal{S}^j)$ denotes the group of sections of the sheaf $\mathcal{S}^j$ for a subset U open in X (see also §1). Then σ is called a quasi-isomorphism, if $\mathsf{H}^j(\sigma)$ is an isomorphism for each j.

We consider complexes bounded below, that is there exists j_0 such that $\mathcal{S}^j = 0$ for each $j < j_0$.

A mapping $\sigma : \mathcal{S}^* \to \mathcal{T}^*$ is called an injective resolution of $\mathcal{S}^*$ if $\mathcal{T}^*$ is a complex of $\mathcal{G}$-modules bounded below, σ is a quasi-isomorphism and the sheaves $\mathcal{T}^b$ are injective, which means that $Hom(\mathcal{B}, \mathcal{T}^b) \to Hom(\mathcal{K}, \mathcal{T}^b)$ is surjective for each injective mapping $\mathcal{K} \to \mathcal{B}$ of sheaves of $\mathcal{G}$-modules.

Let $\mathcal{G}$ be a constant sheaf of rings, may be twisted over $\{i_0, ..., i_{2^r-1}\}$. Suppose that $\mathcal{S}^*$ is a complex of $\mathcal{G}$-modules bounded below. The hypercohomology group ${}_h\mathsf{H}^b(X, \mathcal{S}^*)$ is defined to be the $\mathcal{G}$-module such that

${}_h\mathsf{H}^b(X, \mathcal{S}^*) := [ker(\Gamma(X, \mathcal{T}^b) \to \Gamma(X, \mathcal{T}^{b+1})]/[im(\Gamma(X, \mathcal{T}^{b-1}) \to \Gamma(X, \mathcal{T}^b)]$.
If $\sigma : \mathcal{S}^* \to \mathcal{F}^*$ is a quasi-isomorphism, then σ induces an isomorphism of the hypercohomology groups:

$\sigma : {}_h\mathsf{H}^b(X, \mathcal{S}^*) \cong {}_h\mathsf{H}^b(X, \mathcal{F}^*)$ (see also [13] and the reference $[EV]$ in it). In view of Lemma 16 [22] the hypercohomology groups ${}_h\mathsf{H}^b(X, \mathcal{S}^*)$ are twisted over $\{i_0, ..., i_{2^r-1}\}$, when $\mathcal{S}^*$ and $\mathcal{G}$ are twisted over $\{i_0, ..., i_{2^r-1}\}$.

11. Proposition. *The sequence 10(3) is an acyclic resolution of the sheaf G_N.*

Proof. Each standard simplex Δ^n with $n \ge 1$ has a C^∞ retraction $\hat{z} : \Delta^n \times [0, 1] \to \{y\}$ into a point y belonging to it.

There exists a C^∞ deformation retraction

(1) $\hat{f} : AG \times [0, 1] \to AG$ supplied by the family of mappings

(2) $\hat{f}_n : (AG)_n \times [0, 1] \to (AG)_{n+1}$, where

(3) $(AG)_n := q_A(\coprod_{j \le n} \Delta^j \times G^{j+1}) \subset AG$ and

(4) $\hat{f}_n(|t_1, ..., t_n, h_0[h_1|...|h_n]|, t) := |\Phi(0, t), \Phi(t_1, t), ..., \Phi(t_n, t), h_0[h_1|...|h_n]|$, where $\Phi : [0, 1]^2 \to [0, 1]$ is defined as the composition $\Phi(x, t) := \phi(\min(1, x + t))$ taking ϕ a smooth nondecreasing function $\phi : [0, 1] \to [0, 1]$ such that $\phi(0) = 0$ and $\phi(1) = 1$.

Then for each C^∞ or H^t_p differentiable mapping $v : AG \to \mathcal{A}_r$ we get

$v \circ \hat{f} \circ (q_n \times id) = v^{n+1} \circ \hat{h}_n$ and $v \circ q_A = v^{n+1}$, where $q_n \times id : (\Delta^n \times G^{n+1}) \times [0,1] \to AG \times [0,1]$, $\hat{h}_n : (\Delta^n \times G^{n+1}) \times [0,1] \to \Delta^{n+1} \times G^{n+2}$ is the smooth mapping given by the formula $\hat{h}_n(t_1, ..., t_n, g_0, g_1, ..., g_n, t) = (\Phi(0,t), \Phi(t_1,t), ..., \Phi(t_n,t), e, g_0, g_1, ..., g_n)$.

At the same time Δ^{n+1} has a C^∞ retraction onto Δ^n for each $n \ge 0$ while the group G is C^∞ or H^t_p differentiable and arcwise connected. Therefore, for each $b > 0$ the cohomology group $\mathsf{H}^b(N, AG_N)$ is trivial (see also §2.2 [3] and §54 below).

12. A sheaf $\mathcal{S}$ on X is called soft if for each closed subset Y of X the restriction map $\mathcal{S}(X) \to \mathcal{S}(Y)$ is a surjection.

12.1. Lemma. *For a C^∞ or H^t_p differentiable group G satisfying conditions 4$(A1, A2, C1, C2)$ the sheaf AG_N is soft.*

Proof. Consider a closed subset Y of N and a section σ_Y of AG_N over Y. In accordance with the definition of a section over a closed subset there exists an open set U and an extension σ_U of σ_Y from Y onto U such that $Y \subset U \subset N$. From the paracompactness of N there exists a neighborhood V of Y such that $cl(V) \subset U$, where $cl(V)$ denotes the closure of V in N. Therefore, the extension σ of σ_Y to a global section of AG_N is provided by the formula $\sigma(x) = \hat{f}(\sigma_U(x), \psi(x))$, where $\hat{f} : AG \times [0,1] \to AG$ is a deformation retraction (see §11) and $\psi : N \to [0,1]$ is either a C^∞ or H^t_p differentiable function equal to 1 on V and equal to 0 on $M \setminus U$.

13. Remark. Let now N be a C^∞ or H^∞ manifold over $\mathcal{A}_r$ and $T(N, G, \pi, \Psi)$ be a tangent bundle with $T = TN$ and the projection $\pi : T \to N$, where connecting mappings $\phi_j \circ \phi_k^{-1}$ for $V_j \cap V_k \ne \emptyset$ of the atlas $At(N) = \{(V_j, \phi_j) : j\}$ of the manifold N are $\mathcal{A}_r$ holomorphic for $1 \le r \le 3$, $\phi_j \circ \phi_k^{-1} \in H^\infty$. Denote by $\mathcal{T}$ the sheaf of germs of smooth sections of T. Then $B\mathcal{T}$ denotes the sheaf associated with a the pre-sheaf assigning to each open subset V of N the group of sections of the natural projection $\coprod_{y \in V} B(\pi^{-1}(y)) \to V$, which are locally of the form

(1) $y \mapsto |t_1(y), ..., t_n(y), [\sigma_1(y)|...|\sigma_n(y)]|$, where $t_1, ..., t_n$ are C^∞ for $r = 1$ or H^∞ for $1 \le r \le 3$ functions and $\sigma_1, ..., \sigma_n$ are C^∞ or H^∞ sections of the vector bundle $T(N, G, \pi, \Psi)$. Using constructions above we define $B^{a+1}\mathcal{T}$ for each $a \in \mathbf{N}$ by induction.

Then $B^{a+1}\mathcal{T}$ is the sheaf associated with the pre-sheaf assigning to an open subset V of N the group of sections of the natural projection

$\coprod_{y \in V} B^{a+1}(\pi^{-1}(y)) \to V$ having the local form (1), where $\sigma_1, ..., \sigma_n$ are sections of $B^a\mathcal{T}$ over V. Similarly we define $AB^a\mathcal{T}$.

If now $T = \Lambda^b T^*N$ is the b-th exterior power of the cotangent bundle of N, then the above construction produces the sheaves $AB^a\mathcal{S}^b_{N,\mathcal{A}_r}$ and $B^{a+1}\mathcal{S}^b_{N,\mathcal{A}_r}$ of $AB^a\mathcal{A}_r$ and $B^{a+1}\mathcal{A}_r$ valued respectively differential C^∞ forms on N, where the index $\mathcal{A}_r$ may be omitted, when the Cayley-Dickson algebra

$\mathcal{A}_r$ is specified. In the equation

(2) $w = \sum_J f_J(z) dx_{b_1,j_1} \wedge dx_{b_2,j_2} \wedge ... \wedge dx_{b_k,j_k}$,

where $f_J : N \to AB^a\mathcal{A}_r$ or $f_J : N \to B^{a+1}\mathcal{A}_r$, $z = (z_1, z_2, ...)$ are local coordinates in N, $z_b = x_{b,0}i_0 + x_{b,1}i_1 + ... + x_{b,2^r-1}i_{2^r-1}$, where $z_b \in \mathcal{A}_r$, $x_{b,j} \in \mathbf{R}$ for each b and every $j = 0, 1, ..., 2^r - 1$, $J = (b_1, j_1; b_2, j_2; ...; b_k, j_k)$.

Since each topological vector space Z over $\mathcal{A}_r$ with $2 \le r \le 3$ has the natural twisted structure $Z = Z_0i_0 \oplus Z_1i_1 \oplus ... \oplus Z_{2^r-1}i_{2^r-1}$ with pairwise isomorphic topological vector spaces $Z_0, ..., Z_{2^r-1}$ over $\mathbf{R}$, then TN and T^*N and Λ^bT^*N have twisted structures, where X^* denotes the space of all continuous $\mathcal{A}_r$ additive and $\mathbf{R}$ homogeneous functionals on X with values in $\mathcal{A}_r$, when $2 \le r \le 3$, while X^* over $\mathbf{C}$ is the usual topologically dual space of continuous $\mathbf{C}$-linear functionals on X. Therefore, due to Proposition 6 $B^a\mathcal{T}$ and $AB^a\mathcal{T}$ have the induced twisted structure for each $a \in \mathbf{N}$.

Each section σ of the sheaf $AB^a\mathcal{S}^k_N$ can be written in the form: $\sigma = |h_0, .., h_n, \sigma_0, ..., \sigma_n|$, where $\sigma_0, ..., \sigma_n$ are smooth differential $B^a\mathcal{A}_r$ valued differential k-forms on V and $\{h_j : j = 0, 1, 2, ...\}$ is a C^∞ smooth partition of unity on V. An addition of differential forms induces the additive group structure. As a multiplication there can be taken the external product of differential forms which is twisted over $\mathcal{A}_r$ for $2 \le r \le 3$.

The group of sections of the sheaf $AB^b\mathcal{S}^k_N$ for an open subset V in N denote by $\Gamma(V, AB^b\mathcal{S}^k_N)$. The sequence of the groups $0 \to \Gamma(V, \mathcal{S}^k_N) \to \Gamma(V, A\mathcal{S}^k_n) \to \Gamma(V, B\mathcal{S}^k_N) \to 0$ is exact for each open subset V in N, since the sequence of vector bundles $0 \to \Lambda^kT^*N \to A\Lambda^kT^*N \to B\Lambda^kT^*N \to 0$ is exact.

The twisted structure of the groups $AB^b\mathcal{S}^k_N$ induces the twisted structure of $\Gamma(V, AB^b\mathcal{S}^k_N)$. Therefore, the sequence of sheaves $0 \to B^b\mathcal{S}^k_N \to AB^b\mathcal{S}^k_N \to B^{b+1}\mathcal{S}^k_N \to 0$ is exact as well as for each $b \ge 0$.

The composition of these sequences induces a long exact sequence

(2) $0 \to \mathcal{S}^k_N \to A\mathcal{S}^k_N \xrightarrow{\sigma} AB\mathcal{S}^k_N \xrightarrow{\sigma} ... \xrightarrow{\sigma} AB^b\mathcal{S}^k_N \xrightarrow{\sigma} ...$,

where $\sigma : AB^b\mathcal{S}^k_N \to AB^{b+1}\mathcal{S}^k_N$ is the composition of mappings $AB^b\mathcal{S}^k_N \to B^{b+1}\mathcal{S}^k_N \to AB^{b+1}\mathcal{S}^k_N$. The sequence (2) will be called the bar resolution of the sheaf $\mathcal{S}^k_N$.

Let $\mathcal{S}$ be an arbitrary twisted sheaf on a topological space X. Denote by $A\mathcal{S}$ and $B\mathcal{S}$ sheaves associated with the pre-sheaves $V \mapsto A(\Gamma(V, \mathcal{S}))$ and $V \mapsto B(\Gamma(V, \mathcal{S}))$ correspondingly. The stalks of $A\mathcal{S}$ and $B\mathcal{S}$ are $A\mathcal{S}_x$ and $B\mathcal{S}_x$ at x, while the sequence

(3) $e \to \mathcal{S}_x \to A(\mathcal{S}_x) \to B(\mathcal{S}_x) \to e$

is exact, consequently, the sequence of sheaves

$e \to \mathcal{S} \to A\mathcal{S} \to B\mathcal{S} \to e$

is also exact. The composition of these sequences gives the bar resolution of $\mathcal{S}$

(4) $e \to \mathcal{S} \to A\mathcal{S} \xrightarrow{\sigma} AB\mathcal{S} \xrightarrow{\sigma} ... \xrightarrow{\sigma} AB^b\mathcal{S} \xrightarrow{\sigma} ....$

The complex of sheaves

(5) $\mathcal{B}^*(\mathcal{S}) : A\mathcal{S} \xrightarrow{\sigma} AB\mathcal{S} \xrightarrow{\sigma} ... \xrightarrow{\sigma} AB^b\mathcal{S} \xrightarrow{\sigma} ...$

is called the bar complex of $\mathcal{S}$. The bar resolution of $\mathcal{S}$ is an acyclic resolution of $\mathcal{S}$ that is deduced analogously to the proofs of Proposition 11 and Lemma 12.1. Thus the cohomology of $\mathcal{S}$ is equal to the cohomology of the cochain complex

(6) $\Gamma(N, A\mathcal{S}) \xrightarrow{\sigma} \Gamma(N, AB\mathcal{S}) \xrightarrow{\sigma} ... \xrightarrow{\sigma} \Gamma(N, AB^b\mathcal{S}) \xrightarrow{\sigma} ....$

The complex (6) will be called the bar cochain complex of $\mathcal{S}$ and will be denoted by $C^*_B(\mathcal{S})$.

Each short exact sequence of sheaves $e \to E \to F \to Y \to e$ twisted over generators $\{i_0, i_1, ..., i_{2^r-1}\}$, $2 \le r \le 3$, induces a short exact sequence of complexes sheaves $e \to \mathcal{B}^*(E) \to \mathcal{B}^*(F) \to \mathcal{B}^*(Y) \to e$, where $\mathcal{B}^*(F) : AF \xrightarrow{\sigma} ABF \xrightarrow{\sigma} AB^2F \xrightarrow{\sigma} ... \xrightarrow{\sigma} AB^bF \xrightarrow{\sigma} ...$ is the bar complex of F.

14. Proposition. *If a sequence of groups $e \to K \to G \to J \to e$ is exact, where E and K, G, J are arcwise connected, then the sequence $e \to (W^ME; N, K, \mathbf{P})_{t,H} \to (W^ME; N, G, \mathbf{P})_{t,H} \to (W^ME; N, J, \mathbf{P})_{t,H} \to e$ is exact.*

Proof. In view of Proposition 7.1 [22] $(W^ME; N, K, \mathbf{P})_{t,H}$ is the principal fiber bundle over $(W^MN)_{t,H}$ with the structure group K^k,

$\pi_{K,*} : (W^ME; N, K, \mathbf{P})_{t,H} \to (W^MN)_{t,H}$,

$\pi^{-1}_{K,*} < w_0 >_{t,H} = < w_0 >_{t,H} \times K^k = e \times K^k$, where $e \in (W^MN)_{t,H}$ denotes the unit element. Since the sequence $e \to K^k \to G^k \to J^k \to e$ is exact as well, then the corresponding sequence of wrap groups is exact.

15. Proposition. *Let G be a C^∞ or H^t_p differentiable twisted group over $\{i_0, i_1, ..., i_{2^r-1}\}$ satisfying Conditions 4$(A1, A2, C1, C2)$. Then for each C^∞ or H^t_p principal G -bundle $E(N, G, \pi, \Psi)$ there exists a C^∞ or H^t_p differentiable mapping $\phi : N \to BG$ such that $E \to N$ is the pull-back of the universal principal G bundle by ϕ.*

Proof. Consider an open covering $\mathcal{V} = \{V_j : j \in J\}$ of N, where J is a set, such that for each $j \in J$ there exists a trivialization $\psi_j : \pi^{-1}(V_j) \to V_j \times G$. Define a mapping $g_j : E \to G$ by the formula $g_j(x) = pr_2(\psi_j(x))$ for $x \in \pi^{-1}(V_j)$, $g_j(x) = e$ for $x \notin \pi^{-1}(V_j)$, where e denotes the neutral element in G and $pr_2 : V_j \times G \to G$ is the projection on the second factor.

For a principal G bundle $E(N, G, \pi, \Psi)$ consider a family of $H^{t'}_p$ transition functions $\{g_{i,j} : i, j \in J\}$ related with an open covering $\mathcal{V} := \{V_j : j \in J\}$ of an $H^{t'}_p$ manifold N over $\mathcal{A}_r$, where J is a set, $g_{i,j} : V_i \cap V_j \to \mathcal{A}^*_r$, when the intersection $V_i \cap V_j \ne \emptyset$ is non-void, $1 \le r \le 3$, $n \in \mathbf{N}$. Introduce the mapping

(1) $g_{E,N}(x) := |f_{j(0)}, f_{j(1)}, ...f_{j(n)}, [g_{j(0),j(1)}|g_{j(1),j(2)}|...|g_{j(n-1),j(n)}]|$

such that $g_{E,N} : N \to BG$, where $\{f_j : j \in J\}$ is an $H^{t_1}_p$ partition of

unity subordinated to $\mathcal{U}$ with $t' \le t_1 \le \infty$. Therefore, $g_{E,N}$ can be chosen of the smoothness class $H_p^{t'}$. Thus, $E(N,G,\pi,\Psi)$ is the pull-back of the universal bundle $AG(BG,G,\pi_B^A,\Psi^A)$ by the classifying mapping $g_{E,N}$, where $\pi_B^A : AG \to BG$ is as in §4. Show it in details.

Take a partition of unity of class C^∞ or H_p^t subordinated to to the covering $\mathcal{V}$ and $\Phi : A \to AG$ be the following mapping

$\Phi(y) := |f_{j_0}(\pi(y)), f_{j_1}(\pi(y)), ..., f_{j_n}(\pi(y)), g_{j_0}(y), g_{j_1}(y), ..., g_{j_n}(y)|$, where $j_0, ..., j_n$ are indices such that $f_j(\pi(y)) \ne 0$ for each $j \in \{j_0, ..., j_n\}$. Then Φ is G equivariant, which means that $\Phi(yh) = \Phi(y)h$ for all y and $h \in G$, since $g_j(yh) = pr_2(\psi_j(yh))$ for $yh \in \pi^{-1}(V_j)$ and $g_j(yh) = e$ for $yh \notin \pi^{-1}(V_j)$. Indeed, $y \in \pi^{-1}(y)$ is equivalent to $yh \in \pi^{-1}(V_j)$ for each $h \in G$, since $\pi^{-1}(V_j) = V_j \times G$, where $y = (u,q)$ with $u \in N$ and $q \in G$ and $(u,q)h = (u,qh)$ in local coordinates. Thus $g_j(yh) = g_j(y)R_h$, where $R_h = h$ for $y \in \pi^{-1}(V_j)$ and $R_h = e$ for $y \notin \pi^{-1}(V_j)$.

Therefore, Φ induces a morphism of principal G-bundles

$$\begin{array}{ccc} A & \xrightarrow{\Phi} & AG \\ \downarrow \pi & & \downarrow \\ N & \xrightarrow{\phi} & BG \end{array}$$

where the restriction of ϕ to V_j is $\phi(x)|_{V_j} = |f_{j_0}, f_{j_1}(x), ..., f_{j_n}(x), [g_{j_0}(\sigma(x)) : g_{j_1}(\sigma(x)) : ... : g_{j_n}(\sigma(x))]|$, $\sigma : V_j \to \pi^{-1}(V_j)$ is a smooth section of the restriction $\pi^{-1}(V_j) \to V_j$ for $\pi : E \to N$. Consider equivalence classes $q_j \sim g_j$ if and only if there exist $s_1, ..., s_m \in G$ such that $(s_m(s_{m-1}...(s_1(q_j)...) = g_j$, hence $q_jh \sim g_jh$, since $q_jh \sim g_jh$ if and only if $h^{-1}q_j \sim h^{-1}g_j$, which is equivalent with $h(s_m(s_{m-1}...(s_1(h^{-1}q_j)...) = g_j$. Due to the alternativity of the group G we get $[(...(g_j^{-1}s_1^{-1})...s_{m-1}^{-1})s_m^{-1}][(s_m(s_{m-1}...(s_1g_l)...)] = g_j^{-1}g_l$.

Therefore, in the non-homogeneous coordinates the mapping ϕ takes the form $\phi(x) = |f_{j_0}(x), f_{j_1}(x), ..., f_{j_n}(x), [g_{j_0,j_1}(x)|g_{j_1,j_2}(x)|...|g_{j_{n-1},j_n}(x)]|$, where $g_{j,l}(x) = [g_j(\sigma(x))]^{-1}g_l(\sigma(x))$ are transition functions associated with the open covering of N by the open sets $\{x \in N : f_j(x) > 0\}$. Then the mapping $\phi(x)$ is independent from the choice of σ, since g_j are G-equivariant. All functions g_j are either C^∞ or H_p^t, hence ϕ is either C^∞ or H_p^t correspondingly.

16. Corollary. *For each smooth C^∞ or H_p^t principal $B^b\mathcal{A}_r^*$ bundle with $1 \le r \le 3$ there exists a C^∞ or H_p^t differentiable mapping $\phi : N \to B^{b+1}\mathcal{A}_r^*$ such that $E(N,G,\pi,\Psi)$ is the pull-back of the universal principal $B^b\mathcal{A}_r^*$-bundle by ϕ, where $G = B^b\mathcal{A}_r^*$.*

17. Lemma. *Let G be a differentiable (topological) group twisted over $\{i_0, ..., i_{2^r-1}\}$ for $1 \le r \le 3$ satisfying Conditions 4(A1, A2, C1, C2). Then the group of isomorphism classes of C^∞ smooth (continuous) principal G-bundles over N is isomorphic with the group $[N,BG]^\infty$ of smooth (or $[N,BG]^0$*

of continuous respectively) homotopy classes of smooth (continuous) maps from N to BG.

Proof. Each principal G-bundle over N has properties $4(A1, A2, C1, C2)$ induced by that of G, where locally $\pi^{-1}(V_j) = V_j \times G$, $conj(y) = (u, conj(g))$ for each $y = (u, g) \in V_j \times G$, while $\{i_0, ..., i_{2^r-1}\}$ for $2 \le r \le 3$ is the multiplicative group, which is associative for $r = 2$ and alternative for $r = 3$.

Consider a short exact sequence $e \to G_N \to AG_N \to BG_N \to e$. In view of Lemma 16 [22] it induces the cohomology long exact sequence

$... \to C^\infty(N, AG) \xrightarrow{\pi_*} C^\infty(N, BG) \to \mathsf{H}^1(N, G_N) \to \mathsf{H}^1(N, AG_N) \to ....$

From $\mathsf{H}^1(N, AG_N) \cong e$ we get the isomorphism $C^\infty(N, BG)/\pi_* C^\infty(N, AG) \cong \mathsf{H}^1(N, G_N)$. Then the image $\pi_* C^\infty(N, AG)$ of the group $C^\infty(N, AG)$ in $C^\infty(N, BG)$ consists of all smooth maps from N to BG for which there exist lift mappings from N to AG.

On the other hand, $f \in C^\infty(N, BG)$ has a lift $F : N \to AG$ if and only if f is smooth (or continuous) homotopic to a constant mapping, since $[g_0 : ... : g_n]$ in $(BG)_n$ is the equivalence class $\{(g_0, ..., g_n) \sim (s_m(...(s_1 g_0)...), ..., (s_m...(s_1 g_n)...)) : s_1, ..., s_m \in G, m \in \mathbf{N}\}$, consequently, $C^\infty(N, BG)/\pi_* C^\infty(N, AG) \cong [N, BG]^\infty$. In the class of continuous mappings we get analogously $C^0(N, BG)/\pi_* C^0(N, AG) \cong [N, BG]^0$.

18. Notes. In view of §§4-6 there exists a short exact sequence

$e \to G \to AG \to BG \to e$

of $H_p^{t'}$ homomorphisms due to the twisted structures of G, AG and BG (see Equations $4(A2)$ and $6(8, 9)$).

To groups AG and BG are assigned simplicial topological groups $AG_.$ and $BG_.$ with face homomorphisms $\partial_j : AG_n \to AG_{n-1}$ given by:

(1) $\partial_j(h_0[h_1|...|h_n]) = h_0 h_1[h_2|...|h_n]$ for $j = 0$,

$\partial_j(h_0[h_1|...|h_n]) = h_0[h_1|...|h_j h_{j+1}|...|h_n]$ for $0 < j < n$,

$\partial_j(h_0[h_1|...|h_n]) = h_0[h_1|...|h_{n-1}]$ for $j = n$. While $\partial_j : BG_n \to BG_{n-1}$ has the form:

(2) $\partial_j([h_1|...|h_n]) = [h_2|...|h_n]$ for $j = 0$,

$\partial_j([h_1|...|h_n]) = [h_1|...|h_j h_{j+1}|...|h_n]$ for $0 < j < n$,

$\partial_j([h_1|...|h_n]) = [h_1|...|h_{n-1}]$ for $j = n$.

The degeneracy homomorphisms $s_j : AG_n \to AG_{n+1}$ are prescribed by the formula:

(3) $s_j(h_0[h_1|...|h_n]) = h_0[e|h_1|...|h_n]$ for $j = 0$,

$s_j(h_0[h_1|...|h_n]) = h_0[h_1|...|h_j|e|h_{j+1}|...|h_n]$ for $0 < j < n$,

$s_j(h_0[h_1|...|h_n]) = h_0[h_1|...|h_n|e]$ for $j = n$. While $s_j : BG_n \to BG_{n+1}$ is given by:

(4) $s_j([h_1|...|h_n]) = [e|h_1|...|h_n]$ for $j = 0$,

$s_j([h_1|...|h_n]) = [h_1|...|h_j|e|h_{j+1}|...|h_n]$ for $0 < j < n$,

$s_j([h_1|...|h_n]) = [h_1|...|h_n|e]$ for $j = n$.

Analogous mappings are for simplices:

(5) $\partial^j(t_0, ..., t_{n+1}) = (t_0, ..., t_j, t_j, t_{j+1}, ..., t_{n+1})$ and

(6) $s^j(t_0, ..., t_{n+1}) = (t_0, ..., t_j, \hat{t}_{j+1}, t_{j+2}, ..., t_{n+1})$, where $\hat{t}_{j+1}$ means that t_{j+1} is absent.

The geometric realization $|AG_.|$ of the simplicial space $AG_.$ is defined to be the quotient space of the disjoint union $\sqcup_{n=0}^{\infty} \Delta^n \times G^{n+1}$ by the equivalence relations

(7) $(\partial^j x, \bar{g}) \sim (x, \partial_j \bar{g})$ for each $(x, \bar{g}) \in \Delta^{n-1} \times G^{n+1}$, while $(s^j x, \bar{g}) \sim (x, s_j \bar{g})$ for each $(x, \bar{g}) \in \Delta^{n+1} \times G^{n+1}$. At the same time the geometric realization $|BG_.|$ of the simplicial space $BG_.$ is the quotient of the disjoint union $\sqcup_{n=0}^{\infty} \Delta^n \times G^n$ by the equivalence relations

(8) $(\partial^j x, \bar{g}) \sim (x, \partial_j \bar{g})$ for each $(x, \bar{g}) \in \Delta^{n-1} \times G^n$, while $(s^j x, \bar{g}) \sim (x, s_j \bar{g})$ for each $(x, \bar{g}) \in \Delta^{n+1} \times G^n$.

Consider a non-commutative sphere $\mathcal{C}_r := \{z \in \mathcal{I}_r : |z| = 1\}$ for $r = 2, 3$, where $\mathcal{I}_r := \{z \in \mathcal{A}_r : Re(z) = 0\}$. For $r = 1$ put $\mathcal{C}_r = \{i, -i\}$, where $i = (-1)^{1/2}$. Let $\mathbf{Z}(\mathcal{C}_r)$ denotes the additive group $\mathbf{Z}^{\mathcal{C}_r}/\mathcal{Z}$, where $\mathbf{Z}^{\mathcal{C}_r} := \prod_{b \in \mathcal{C}_r} T_b$, $T_b = \mathbf{Z}b$ for each $b \in \mathcal{C}_r$, $\mathbf{Z}$ is the additive group of integers, $\mathcal{Z}$ is the equivalence relation such that $T_b \times T_{-b}/\mathcal{Z} = T_b$ for each $b \in \mathcal{C}_r$. For $2 \le r \le 3$ the group $\mathbf{Z}(\mathcal{C}_r)$ is isomorphic with $\mathbf{Z}^{\alpha}$, where $card(\alpha) = card(\mathbf{R}) =: \mathsf{c}$. Particularly, $\mathbf{Z}(\mathcal{C}_1) = \mathbf{Z}i$ for $r = 1$.

Henceforth, we consider twisted sheaves and cohomologies over $\{i_0, ..., i_{2^r-1}\}$, where $2 \le r \le 3$. In particular, the complex case will also be included for $r = 1$, but the latter case is commutative over $\mathbf{C}$. So we can consider simultaneously $1 \le r \le 3$ and generally speak about twisting undermining that for $r = 1$ it is degenerate.

19. Proposition. *Let G be the group either $\mathcal{A}_r^*$ or $\mathbf{Z}(\mathcal{C}_r)$, where $1 \le r \le 3$. Then for each H^{∞} smooth manifold N over $\mathcal{A}_r$ and each $b \ge 2$ the group $\mathsf{H}^b(N, \mathbf{Z}(\mathcal{C}_r))$ is isomorphic with:*

(1) *the group $\mathsf{E}(N, B^{b-2}G)$ of isomorphism classes of smooth principal $B^{b-2}G$-bundles over N;*

(2) *the group $[N, B^{b-1}G]^{\infty}$ of smooth homotopy classes of smooth mappings from N to $B^{b-1}G$.*

Proof. In view of Corollary 3.4 [27, 28] there exists the short exact sequence

(1) $0 \to \mathbf{Z}(\mathcal{C}_r) \xrightarrow{\eta} \mathcal{A}_r \to \mathcal{A}_r^* \to 1$,

since $\exp(M + 2\pi k M/|M|) = \exp(M)$ for each non-zero purely imaginary $M \in \mathcal{I}_r$ (with $Re(M) = 0$) and every $k \in \mathbf{Z}$, $1 \le r \le 3$, where $\eta(z) = 2\pi z$ for each $z \in \mathcal{A}_r$. If $f : \mathcal{A}_r \to \mathcal{A}_r^*$ is a differentiable function, then $(dLnf).h = w(h)$ is the differential one-form considering d as the external differentiation over $\mathbf{R}$, where $h \in \mathcal{A}_r$. In the particular case of $G = \mathcal{A}_r^*$ with $1 \le r \le 3$ there exist further short exact sequences

(2) $1 \to \mathcal{A}_r^* \to A\mathcal{A}_r^* \to B\mathcal{A}_r^* \to 1$
(3) $1 \to B\mathcal{A}_r^* \to AB\mathcal{A}_r^* \to B^2\mathcal{A}_r^* \to 1$
(4) $1 \to B^m\mathcal{A}_r^* \to AB^m\mathcal{A}_r^* \to B^{m+1}\mathcal{A}_r^* \to 1$.

Therefore, identifying the ends of these short exact sequences we get the long exact sequence

(5) $0 \to \mathbf{Z}(\mathcal{C}_r) \to \mathcal{A}_r \to A\mathcal{A}_r^* \to AB\mathcal{A}_r^* \to ... \to AB^m\mathcal{A}_r^* \to ...$,
where $\sigma : \mathcal{A}_r \to A\mathcal{A}_r^*$,..., $\sigma : AB^{m-1}\mathcal{A}_r^* \to AB^m\mathcal{A}_r^*$ are homomorphisms, all terms $\mathcal{A}_r$, $A\mathcal{A}_r^*$,...,$AB^m\mathcal{A}_r^*$,... are contractible spaces.

Suppose now that N and E are of class H^∞. Let $\mathsf{C}^\infty(N, AB^m\mathcal{A}_r^*)$ denotes the sheaf of germs of C^∞ functions from N into $AB^m\mathcal{A}_r^*$. Thus, we get the functor C^∞. Then the application of C^∞ functor to the long exact sequence (5) gives:

(6) $0 \to \mathbf{Z}(\mathcal{C}_r)_N \to \mathsf{C}^\infty(N, \mathcal{A}_r) \to \mathsf{C}^\infty(N, A\mathcal{A}_r^*) \to \mathsf{C}^\infty(N, AB\mathcal{A}_r^*) \to ... \to \mathsf{C}^\infty(N, AB^m\mathcal{A}_r^*) \to ...$,
where $\sigma_* : \mathsf{C}^\infty(N, \mathcal{A}_r) \to \mathsf{C}^\infty(N, A\mathcal{A}_r^*)$,...,$\sigma_* : \mathsf{C}^\infty(N, AB^{m-1}\mathcal{A}_r^*) \to \mathsf{C}^\infty(N, AB^m\mathcal{A}_r^*)$ are induced homomorphisms.

The latter exact sequence is called the bar resolution of $\mathbf{Z}(\mathcal{C}_r)_N$. Sheaves $\mathsf{C}^\infty(N, \mathcal{A}_r)$ and $\mathsf{C}^\infty(N, AB^m\mathcal{A}_r^*)$ are contractible, since $\mathcal{A}_r$ and $AB^m\mathcal{A}_r^*$ are contractible. Therefore, the cohomology of the sheaf $\mathbf{Z}(\mathcal{C}_r)_N$ can be computed using the complex

(7) $C^\infty(N, \mathcal{A}_r) \to C^\infty(N, A\mathcal{A}_r^*) \to ... \to C^\infty(N, AB^m\mathcal{A}_r^*) \to ...$
with homomorphisms
$\sigma_* : C^\infty(N, \mathcal{A}_r) \to C^\infty(N, A\mathcal{A}_r^*)$,...,$\sigma_* : C^\infty(N, AB^{m-1}\mathcal{A}_r^*) \to C^\infty(N, AB^m\mathcal{A}_r^*)$.
The long exact sequence (7) we call a bar cochain complex of $\mathbf{Z}(\mathcal{C}_r)_N$. The cohomology of $\mathbf{Z}(\mathcal{C}_r)_N$ computed with the help of the bar complex is denoted by $\mathsf{H}_b^*(N, \mathbf{Z}(\mathcal{A}_r)_N)$ and it is called the bar cohomology of $\mathbf{Z}(\mathcal{C}_r)_N$. Then $\pi_0 C^\infty(N, B\mathcal{A}_r^*)$ is the first bar cohomology $\mathsf{H}_b^1(N, \mathbf{Z}(\mathcal{A}_r)_N)$ of $\mathbf{Z}(\mathcal{C}_r)_N$.

For the generalized exponential sequence

$0 \to \mathbf{Z}(\mathcal{C}_r)_N \to AB^{<b-2}G_N \to B^{b-2}G_N[2-b] \to e$ there exists the cohomology long exact sequence

$... \to {}_h\mathsf{H}^{b-1}(N, AB^{<b-2}G_N) \to \mathsf{H}^1(N, B^{b-2}G_N)$
$\to \mathsf{H}^b(N; \mathbf{Z}(\mathcal{C}_r)) \to {}_h\mathsf{H}^b(N, AB^{<b-2}G_N) \to ...$,
where $AB^{<b}G_N$ is the complex

$K_N \xrightarrow{\sigma} AG_N \xrightarrow{\sigma} ABG_N \xrightarrow{\sigma} AB^2G_N \xrightarrow{\sigma} ... \xrightarrow{\sigma} AB^{b-1}G_N$,
K is equal to either $\mathcal{A}_r$ or $A\mathbf{Z}(\mathcal{C}_r)$ for $G = \mathcal{A}_r^*$ or $G = B\mathbf{Z}(\mathcal{C}_r)$ respectively, where ${}_h\mathsf{H}^b(N, \mathcal{B})$ denotes a hypercohomology on N with coefficients in a complex of sheaves $\mathcal{B}$ (see its definition in [3, 12, 13] and §10 above).

We have that for each $b \geq 0$ the sheaf AB^bG_N is acyclic. Consider the groups of global sections $AB^bG_N(N)$ of the sheaves AB^bG_N. Therefore, the cohomology of the complex $AB^{<b-2}G_N$ is equal to the cohomology of the cochain complex

$K \xrightarrow{\sigma} AG_N(N) \xrightarrow{\sigma} ABG_N(N) \xrightarrow{\sigma} AB^2G_N(N) \xrightarrow{\sigma} ... \xrightarrow{\sigma} AB^{b-1}G_N(N)$.

Thus, ${}_h\mathsf{H}^m(N, AB^{<b-2}G_N) \cong \mathsf{H}^m(N, AB^{<b-2}G_N(N)) \cong e$ for each $m > b-2$, consequently, the coboundary homomorphism $\mathsf{H}^1(N, B^{b-2}G_N) \to \mathsf{H}^b(N; \mathbf{Z}(\mathcal{C}_r))$ is an isomorphism (see also Chapter 2 §4 in [3] for abelian sheafs).

The second statement of this proposition follows from Lemma 17.

20. Lemma. *Let X be a topological vector space over $\mathcal{A}_r$, $2 \le r \le 3$. Then AX and BX with respect to the additive group structure of X and with respect to the multiplication on scalars from $\mathcal{A}_r$ in homogeneous coordinates are $\mathcal{A}_r$ vector spaces and the projection $AX \to BX$ is $\mathbf{R}$-homogeoneous and $\mathcal{A}_r$ additive.*

Proof. Define the multiplications by: for $\mathcal{A}_r \times AX \to AX$ as

$s|t_1, ..., t_n; v_0[v_1|...|v_n]| = |t_1, ..., t_n; sv_0[sv_1|...|sv_n]|$,

for $AX \times \mathcal{A}_r \to AX$ as

$|t_1, ..., t_n; v_0[v_1|...|v_n]|s = |t_1, ..., t_n; v_0s[v_1s|...|v_ns]|$,

for $\mathcal{A}_r \times BX \to BX$ as

$s|t_1, ..., t_n; [v_1|...|v_n]| = |t_1, ..., t_n; [sv_0|...|sx_n]|$,

for $BX \times \mathcal{A}_r \to BX$ as

$|t_1, ..., t_n; [v_1|...|v_n]|s = |t_1, ..., t_n; [v_1s|...|v_ns]|$.

Then if $q_j = s_m(s_{m-1}...(s_1v_j)...)$ for each j, then for $z \ne 0$ we get $zq_j = z(s_m...(s_1(z^{-1}(zv_j)))...)$ due to the alternativity of the octonion algebra $\mathbf{O}$, while for $z = 0$ we trivially get $0 = (s_m...(s_10)...)$. Thus such multiplication is compatible with the equivalence relations, since $X = X_0i_0 \oplus ... \oplus X_{2^r-1}i_{2^r-1}$, where $X_0, ..., X_{2^r-1}$ are pairwise isomorphic topological vector spaces over $\mathbf{R}$ such that we put $vx = xv$ for each $v \in X_j$ and $x \in \mathbf{R}$.

Since $\mathbf{R}$ is the center of the algebra $\mathbf{O}$, then the projection from AX to BX is $\mathbf{R}$-linear. Evidently, it is additive as the additive group homomorphism.

21. Remark. Let $N_{\cdot}$ be a simplicial smooth manifold over $\mathcal{A}_r$, where $0 \le r \le 3$. A smooth m-form w on the geometric realization $|N_{\cdot}|$ of $N_{\cdot}$ is defined to be as a family $\{w^k : k\}$ of smooth differential m-forms w^k on $\Delta^k \times N_k$ with values in $\mathcal{A}_r$ being applied to vectors, satisfying for each $0 \le j \le n$ the compatibility conditions:

(1) $(\partial^j \times id)^* w^n = (id \times \partial_j)^* w^{n-1}$

(2) $(s^j \times id)^* w^n = (id \times s_j)^* w^{n+1}$,

where $\partial^j \times id$, $id \times \partial_j$, $s^j \times id$ and $id \times s_j$ are the maps as follows:

(3) $id \times \partial_j : \Delta^{n-1} \times N_n \to \Delta^{n-1} \times N_{n-1}$, $\partial^j \times id : \Delta^{n-1} \times N_n \to \Delta^n \times N_n$, $id \times s_j : \Delta^{n+1} \times N_n \to \Delta^{n+1} \times N_{n+1}$, $s^j \times id : \Delta^{n+1} \times N_n \to \Delta^n \times N_n$ such that ∂^j and s^j are coface and the codegeneracy maps on Δ^n and ∂_j, s_j are the face and the degeneracy maps on N_n. We consider w taking values in a vector space or an algebra over $\mathcal{A}_r$ as is specified below.

For a Lie group G either over $\mathbf{R}$ or may be twisted over $\mathcal{A}_r$ and its Lie algebra g put $g^{-1}dg$ as the canonical g-valued connection 1-form on G (see also Lemma 20). Under the mapping $g \mapsto hg$ and $dg \mapsto hdg$ we have $(g^{-1}h^{-1})(hdg) = g^{-1}dg$ due to the alternativity of G and the Maufang identity $(xy)(zx) = x(yz)x$ for each $x, y, z \in \mathbf{O}$ and $de = d(g^{-1}g) = 0 = (dg^{-1})g + g^{-1}dg = [(dg^{-1})h^{-1}](hg) + (g^{-1}h^{-1})(hdg)$. Iterating this relation due to the alternativity of $\mathbf{O}$ and the Moufang identities in it we get the equivariance condition in homogeneous coordinates over $\mathbf{O}$ as well: $[(...(g^{-1}s_1^{-1})...s_{m-1}^{-1})s_m^{-1}][s_m(s_{m-1}...(s_1dg_1)...)] = g_1^{-1}dg_1$.

The total space AG of the universal principal g-bundle $AG \to BG$ carries a smooth g-valued form w. The evaluation of w is $w|x_0, ..., x_n, g_0, ..., g_n| = x_0g_0^{-1}dg_0 + x_1g_1^{-1}dg_1 + ... + x_ng_n^{-1}dg_n$, where $x_0, ..., x_n$ are barycentric coordinates in Δ^n. Each term $x_ng_n^{-1}dg_n.s$ is in g for each $s \in \mathsf{g}$ such that $g_j^{-1}dg_j = \pi_j^*(g^{-1}dg|_{T_{g_j}G})$, where $\pi_j : G^{n+1} \to G$ is the projection on the j-th factor and $g^{-1}dg|_{T_{g_j}G}$ is the restriction of $g^{-1}dg$ to the tangent space $T_{g_j}G$ of G at g_j.

For $A\mathcal{A}_r^*$ with $2 \le r \le 3$ define the canonical connection 1-form $A(z^{-1}dz)$ by the family of $A\mathcal{A}_r$-valued 1-forms $A(z^{-1}dz)^n$ on $\Delta^n \times (\mathcal{A}_r^*)^{n+1}$ such that $A(z^{-1}dz)^n$ evaluated on a vector $(v_0, ..., v_n)$ at a point $|t_1, ..., t_n, z_0[z_1|...|z_n]|$ is given by the formula

(4) $(A(z^{-1}dz)^n|_{|t_1,...,t_n,z_0[z_1|...|z_n]|}.(v_0, ..., v_n) = |t_1, ..., t_n, z_0^{-1}v_0[z_1^{-1}v_1|...|z_n^{-1}v_n]|$

and formally denote it by

(5) $(A(z^{-1}dz)^n|_{|t_1,...,t_n,z_0[z_1|...|z_n]|} = |t_1, ..., t_n, z_0^{-1}dz_0[z_1^{-1}dz_1|...|z_n^{-1}dz_n]|$.

For $B\mathcal{A}_r^*$ with $2 \le r \le 3$ the canonical connection 1-form $B(z^{-1}dz)$ on $B\mathcal{A}_r^*$ is defined by the family of $B\mathcal{A}_r$-valued 1-forms $B(z^{-1}dz)^n$ on $\Delta^n \times (\mathcal{A}_r^*)^n$, where

(6) $B(z^{-1}dz)^n|_{|t_1,...,t_n,[z_1|...|z_n]|} = |t_1, ..., t_n, [z_1^{-1}dz_1|...|z_n^{-1}dz_n]|$.

We have that

$(\partial^j \times id)^*A(z^{-1}dz)^n|_{|t_1,...,t_{n-1};z_0[z_1|...|z_n]|} =$
$|t_1, ..., t_j, t_j, t_{j+1}, ..., t_{n-1}; z_0^{-1}dz_0[z_1^{-1}dz_1|...|z_n^{-1}dz_n]|$ and
$(\partial^j \times id)^*B(z^{-1}dz)^n|_{|t_1,...,t_{n-1},[z_1|...|z_n]|} =$
$|t_1, ..., t_j, t_j, t_{j+1}, ..., t_{n-1}, [z_1^{-1}dz_1|...|z_n^{-1}dz_n]|$ and
$(id \times \partial_j)^*A(z^{-1}dz)^{n-1}|_{|t_1,...,t_{n-1};z_0[z_1|...|z_n]|} =$
$|t_1, ..., t_{n-1}; (z_0^{-1}dz_0) + (z_1^{-1}dz_1)[z_2^{-1}dz_2|...|z_n^{-1}dz_n]|$ for $j = 0$,
$(id \times \partial_j)^*A(z^{-1}dz)^{n-1}|_{|t_1,...,t_{n-1};z_0[z_1|...|z_n]|} =$
$|t_1, ..., t_{n-1}; z_0^{-1}dz_0[z_1^{-1}dz_1|...|z_{j-1}^{-1}dz_{j-1}|(z_j^{-1}dz_j)+(z_{j+1}^{-1}dz_{j+1})|z_{j+2}^{-1}dz_{j+2}|...|z_n^{-1}dz_n]|$ for $0 < j < n$,
$(id \times \partial_j)^*A(z^{-1}dz)^{n-1}|_{|t_1,...,t_{n-1};z_0[z_1|...|z_n]|} =$
$|t_1, ..., t_{n-1}; z_0^{-1}dz_0[z_1^{-1}dz_1|...|z_{n-1}^{-1}dz_{n-1}]|$ for $j = n$, while
$(id \times \partial_j)^*B(z^{-1}dz)^{n-1}|_{|t_1,...,t_{n-1};[z_1|...|z_n]|} =$

$|t_1,...,t_{n-1};[z_2^{-1}dz_2|...|z_n^{-1}dz_n]|$ for $j=0$,

and $(id\times\partial_j)^*B(z^{-1}dz)^{n-1}|_{|t_1,...,t_{n-1};[z_1|...|z_n]|}=$
$|t_1,...,t_{n-1};[z_1^{-1}dz_1|...|z_{j-1}^{-1}dz_{j-1}|(z_j^{-1}dz_j)+(z_{j+1}^{-1}dz_{j+1})|z_{j+2}^{-1}dz_{j+2}|...|z_n^{-1}dz_n]|$ for $0<j<n$

$(id\times\partial_j)^*B(z^{-1}dz)^{n-1}|_{|t_1,...,t_{n-1};[z_1|...|z_n]|}=$
$|t_1,...,t_{n-1};[z_1^{-1}dz_1|...|z_{n-1}^{-1}dz_{n-1}]|$ for $j=n$
and using the equivalence relations $4(2,3,5)$, we get the compatibility Condition (1) for such differential forms, since ∂^j corresponds to inserting $x_j=0$ and the latter corresponds to $t_j=t_{j+1}$, because $t_j=x_0+...+x_{j-1}$, while $h_j=e$ corresponds to $g_j=g_{j+1}$.

Further we get:

$(s^j\times id)^*A(z^{-1}dz)^n|_{|t_1,...,t_{n+1};z_0[z_1|...|z_n]|}=$
$|t_1,...,t_j,t_{j+2},...,t_{n+1};z_0^{-1}dz_0,[z_1^{-1}dz_1|...|z_n^{-1}dz_n]|$ for each j and

$(s^j\times id)^*B(z^{-1}dz)^n|_{|t_1,...,t_{n+1};[z_1|...|z_n]|}=$

$|t_1,...,t_j,t_{j+2},...,t_{n+1};[z_1^{-1}dz_1|...|z_n^{-1}dz_n]|$,
since $s^j(x_0,...,x_{n+1})=(x_0,...,x_{j-1},x_j+x_{j+1},x_{j+2},...,x_{n+1})$ and $x_j=0$ corresponds to $t_j=t_{j+1}$. Then

$(id\times s_j)^*A(z^{-1}dz)^{n+1}|_{|t_1,...,t_{n+1};z_0[z_1|...|z_n]|}=$
$|t_1,...,t_{n+1};z_0^{-1}dz_0[0|z_1^{-1}dz_1|...|z_n^{-1}dz_n]|$ and

$(id\times s_j)^*A(z^{-1}dz)^{n+1}|_{|t_1,...,t_{n+1};z_0[z_1|...|z_n]|}=$
$|t_1,...,t_{n+1};z_0^{-1}dz_0,[z_1^{-1}dz_1|...|z_j^{-1}dz_j|0|z_{j+1}^{-1}dz_{j+1}|...|z_n^{-1}dz_n]|$ for $0<j<n$ and

$(id\times s_j)^*A(z^{-1}dz)^{n+1}|_{|t_1,...,t_{n+1};z_0[z_1|...|z_n]|}=$
$|t_1,...,t_{n+1};z_0^{-1}dz_0,[z_1^{-1}dz_1|...|z_n^{-1}dz_n|0]|$ for $j=n$ and shortly write

$(id\times s_j)^*B(z^{-1}dz)^{n+1}|_{|t_1,...,t_{n+1};[z_1|...|z_n]|}=$
$|t_1,...,t_{n+1};[z_1^{-1}dz_1|...|z_j^{-1}dz_j|0|z_{j+1}^{-1}dz_{j+1}|...|z_n^{-1}dz_n]|$ for each j.
Using the equivalence relations $4(2,3,5)$ in AB^bG and $B^{b+1}G$ we get the compatibility Condition (2).

For a twisted group G satisfying Conditions $4(A1,A2,C1,C2)$ a smooth k-form on AB^bG and $B^{b+1}G$ is defined by induction. Let smooth differential k-forms on B^bG and each $\Delta^k\times(B^bG)^m$ for $k,m\ge 0$ be defined. Then a smooth k-form w on $AB^{b+1}G$ consists of a family of k-forms w^n on $\Delta^n\times(B^bG)^{n+1}$ satisfying the compatibility conditions $(1,2)$. For $B^{b+1}G$ a smooth k-form w on $B^{b+1}G$ consists of a family of k-forms w^n on $\Delta^n\times(B^bG)^{n+1}$ satisfying the compatibility Conditions $(1,2)$. Then a smooth k-form w on $\Delta^k\times(B^{b+1}G)^m$ consists of a family of k-forms w^n on $\Delta^k\times(\Delta^n\times(B^bG)^{n+1})^m$ satisfying the compatibility conditions:

(7) $id_{\Delta^k}\times(\partial^j\times id)^m)^*w^n=(id_{\Delta^k}\times(id\times\partial_j)^m)^*w^{n-1}$,

(8) $(id_{\Delta^k}\times(s^j\times id)^m)^*w^n=(id_{\Delta^k}\times(id\times s_j)^m)^*w^{n+1}$.

It was shown above that the groups $AB^b\mathcal{A}_r$ and $B^{b+1}\mathcal{A}_r$ also have the structures of $\mathcal{A}_r$ vector spaces. Therefore, the canonical connection 1-form $AB^b(z^{-1}dz)$ on $AB^b\mathcal{A}_r^*$ is a 1-form on $AB^b\mathcal{A}_r^*$ such that it satisfies the inductive formula:

(9) $AB^b(z^{-1}dz)|_{|t_1,...,t_n,g_0[g_1|...|g_n]|} = |t_1,...,t_n, B^b(g_0^{-1}g_0)[B^b(g_1^{-1}dg_1)|...|B^b(g_n^{-1}dg_n)]|$.

Then the canonical connection 1-form $B^{b+1}(z^{-1}dz)$ on $B^{b+1}\mathcal{A}_r^*$ is a 1-form on $B^{b+1}\mathcal{A}_r^*$ such that

(10) $B^{b+1}(z^{-1}dz)|_{|t_1,...,t_n,[g_1|...|g_n]|} = |t_1,...,t_n,[B^b(g_1^{-1}dg_1)|...|B^b(g_n^{-1}dg_n)]|$, where $g_0, g_1, ..., g_n \in B^b\mathcal{A}_r^*$ and $B^b(g_j^{-1}dg_j)$ is the canonical connection 1-form $B^b(z^{-1}dz)$ on $B^b\mathcal{A}_r^*$ evaluated at g_j.

22. Gerbes over quaternions and octonions. Consider twisted groups C, K, G satisfying Conditions $4(A1, A2, C1, C2)$. If

$(CE1)$ $e \to C_0 \to K_0 \to G_0 \to e$ is a topological central extension, then we say, that

$(CE2)$ $e \to C \to K \to G \to e$ is a topological twisted extension.

A gerbe on a topological space X is a sheaf $\mathcal{S}$ of categories satisfying the conditions $(G1 - G3)$:

$(G1)$ for each open subset V in X the category $\mathcal{S}(V)$ is a groupoid, which means that every morphism is invertible;

$(G2)$ each point $x \in X$ has a neighborhood V_x for which $\mathcal{S}(V_x)$ is non-empty;

$(G3)$ any two objects P_1 and P_2 of $\mathcal{S}(V)$ are locally isomorphic, that is, each $x \in V$ has a neighborhood Y for which the restrictions $P_1|_Y$ and $P_2|_Y$ are isomorphic.

A gerbe $\mathcal{S}$ is called bound by a sheaf $\mathcal{G}$ of twisted groups over $\mathcal{A}_r$ satisfying Conditions $5(A1, C1, C2, 7)$, if for each open subset V in X and every object P of $\mathcal{S}(V)$ there exists an isomorphism of sheaves $\nu : Aut(P) \to \mathcal{G}|_V$, where $\mathcal{G}|_V$ denotes the restriction of the sheaf $\mathcal{G}$ onto V, while $Aut(P)$ is the sheaf of automorphisms of P so that for an open subset Y in V the group $Aut(P)(Y)$ is the group of automorphisms of the restriction $s_Y(P)$. It is supposed that such an isomorphism commutes with with morphisms of $\mathcal{S}$ and must be compatible with restrictions to smaller open subsets.

Two gerbes $\mathcal{S}$ and $\mathcal{E}$ bounded by $\mathcal{G}$ on a manifold N are equivalent, if they satisfy $(G4, G5)$:

$(G4)$ if V is an open subset in X, then there exists an equivalence of categories $\mu_V : \mathcal{S}(V) \to \mathcal{E}(V)$ so that for each object P of $\mathcal{S}(V)$ there is a commutative diagram:

$\mu_V : Aut_{\mathcal{S}(V)}(P) \to Aut_{\mathcal{E}(V)}(P)$,
$\nu_{\mathcal{S}} : Aut_{\mathcal{S}(V)}(P) \to \Gamma(V, \mathcal{G})$,
$\nu_{\mathcal{E}} : Aut_{\mathcal{E}(V)}(P) \to \Gamma(V, \mathcal{G})$ such that
$\nu_{\mathcal{S}} = \nu_{\mathcal{E}}(\mu_V)$;

$(G5)$ for each pair of open subsets V and Y in N with $Y \subset V$ there exists an invertible natural transformation: $\beta : R_{\mathcal{E}}(\mu_V) = \mu_Y(R_{\mathcal{S}})$, where $R_{\mathcal{S}} : \mathcal{S}(V) \to \mathcal{S}(Y)$ denotes the natural restriction transformation. It is also imposed the condition, that for a triple of open subsets $Y \subset V \subset J$ in N the compatibility conditions are satisfied.

If there is a principal G-bundle $E(B, G, \pi, \Psi)$ and and an extension $(CE1, CE2)$ of topological groups, then there exists a gerbe $\mathcal{G}_\pi$ bound by C_N on B. This gerbe is constructed from the sheaf of sections of the bundle $E(B, G, \pi, \Psi)$ by posing for each open subset V of B objects and morphisms of $\mathcal{G}_\pi(V)$ as follows. Associate with each section $s : V \to \pi^{-1}(V)$ of $\pi : \pi^{-1}(V) \to V$ the G-equivariant map $t_s : \pi^{-1}(V) \to G$ such that $t_s(z)s(\pi(z)) = z$ for every $z \in \pi^{-1}(V)$. We have as well the pull-back of principal C-bundle $K \to G$ from G to $\pi^{-1}(V)$ due to the mapping $t_s : \pi^{-1}(V) \to G$.

The composition $\pi \circ \pi_s : E_s \to V$ is a principal K-bundle having a lifting of the structure group of $\pi^{-1}(V) \to V$ to K. Then pairs (E, f) of principal K-bundles $\pi_V : E(V, K, \pi, \Psi) \to V$ and principal C-bundles $f : E \to \pi^{-1}(V)$ such that there exists the commutative diagram with $\pi(f(*)) = \pi_V(*)$.

A morphism of principal K-bundles $\eta : E \to E_1$ from (E, f) to (E_1, f_1) is described with the help of the condition $f_1(\eta(*)) = f$ with the corresponding commutative diagram. Therefore, the group of automorphisms of every object (E, f) of $\mathcal{G}_\pi(V)$ is the group of mappings from V to C being the section of the sheaf C_N over V, consequently, $\mathcal{G}_\pi$ is the gerbe bound by C_N.

The constructed above gerbe $\mathcal{G}_\pi$ has a global section if and only if there exists a lifting of the structure group from C to K. For $G = B\mathcal{A}_r^*$ the extension $(CE1, CE2)$ takes the form

$1 \to \mathcal{A}_r^* \to A\mathcal{A}_r^* \to B\mathcal{A}_r^* \to 1$. Then each principal $A\mathcal{A}_r^*$-bundle is trivial, since $A\mathcal{A}_r^*$ is contractible. Hence the gerbe $\mathcal{G}_\pi$ has a global section if and only if $E(N, B\mathcal{A}_r^*, \pi, \Psi)$ is a trivial $B\mathcal{A}_r^*$-bundle.

Construct now another gerbe $\mathcal{L}_\pi$ of local sections of the bundle $E(B, B\mathcal{A}_r^*, \pi, \Psi)$. For each open subset V in B the objects of $\mathcal{L}_\pi(V)$ are sections of E over V so that each local section $s : V \to \pi^{-1}(V)$ induces a $B\mathcal{A}_r^*$-equivariant mapping $t_s : \pi^{-1}(V) \to B\mathcal{A}_r^*$ that induces the mapping $\tau_s = t_s(s(*)) : V \to B\mathcal{A}_r^*$.

If E_s is a principal $\mathcal{A}_r^*$-bundle over V induced by the mapping τ_s, then a morphism between the objects $s, s_1 \in \mathcal{L}_\pi(V)$ induces the morphism $\mathsf{E}_s \to \mathsf{E}_{s_1}$ of the corresponding principal $\mathcal{A}_r^*$-bundles. Then $\mathcal{L}_\pi$ is a gerbe bounded by $(\mathcal{A}_r^*)_N$, where $2 \le r \le 3$. Therefore, the natural transformation $\mathcal{L}_\pi(V) \to \mathcal{G}_\pi(V)$ sending a section s to the pull-back E_s of the universal principal $\mathcal{A}_r^*$-bundle by t_s is an equivalence of categories, which extends to an equivalence of gerbes $\mathcal{L}_\pi \to \mathcal{G}_\pi$.

For a gerbe $\mathcal{G}$ on N bounded by $(\mathcal{A}_r^*)_N$ with $2 \le r \le 3$, assigning to each object Q in $\mathcal{G}(V)$ an $\mathcal{S}^1_{N,\mathcal{A}_r}$-torsor $\mathcal{C}_{\mathcal{O}Q}$ on V induces a connective structure.

This torsor $\mathcal{C}_{\mathcal{O}Q}$ consists of a sheaf on which $\mathcal{S}^1_{N,r}$ acts so that for each point $x \in N$ there exists a neighborhood V having the property that for each open subset $Y \subset V$ the group $\mathcal{C}_{\mathcal{O}Q}(Y)$ is a principal homogeneous space under the group $\Gamma(Y, \mathcal{S}^1_{N,\mathcal{A}_r})$. This assignment $Q \mapsto \mathcal{C}_{\mathcal{O}Q}(V)$ need to be functorial in accordance with restrictions from V onto Y. Moreover, for each morphism $\phi : Q \to J$ of objects of $\mathcal{G}(V)$ there exists an isomorphism $\phi_* : \mathcal{C}_{\mathcal{O}Q}(V) \to \mathcal{C}_{\mathcal{O}J}(V)$ of $\mathcal{S}^1_{N,\mathcal{A}_r}$-torsors. Since $\mathcal{G}$ is a gerbe, then ϕ is an isomorphism and ϕ_* is compatible with compositions of morphisms and with restrictions to smaller open subsets, $Y \subset V$. If ϕ is an automorphism of Q induced by an $\mathcal{A}_r^*$-valued function g we suppose that ϕ_* is an automorphism $\nabla \mapsto \nabla - dLn(g)$ of the $\mathcal{S}^1_{N,\mathcal{A}_r}$-torsor $\mathcal{C}_{\mathcal{O}Q}(V)$.

Consider a connection ω on a smooth principal $B\mathcal{A}_r^*$-bundle $E(N, B\mathcal{A}_r^*, \pi, \Psi)$ and let V be an open subset in N such that $\mathcal{G}_\pi(V)$ is non-void and let ω_V be the restriction of ω to $\pi^{-1}(V)$. To each element (E, f) of $\mathcal{G}_\pi(V)$ it is possible assign a set $\mathcal{C}^\omega_{\mathcal{O}E}(V)$ of connections on E compatible with ω. If $\omega(q(*)) = f^*\omega$ for principal $\mathcal{A}_r^*$-bundles $q : A\mathcal{A}_r \to B\mathcal{A}_r$ and $f : E \to \pi^{-1}(V)$, then ω generates an element $\hat{\omega} \in \mathcal{C}_{\mathcal{O}E}(V)$. Therefore, the assignment $\omega \mapsto \mathcal{C}^\omega_{\mathcal{O}}$ is the connective structure on $\mathcal{G}_\pi$.

The equivalence of gerbes $\mathcal{L}_\pi \to \mathcal{G}_\pi$ implies an extension of a pull-back of the connective structure from $\mathcal{G}_\pi$ to $\mathcal{L}_\pi$.

23. Corollary. *A mapping posing to the isomorphism class of a principal $B\mathcal{A}_r^*$-bundle $E(B, B\mathcal{A}_r^*, \pi, \Psi)$, $\pi : E = A\mathcal{A}_r^* \to B\mathcal{A}_r^*$, the equivalence class of the gerbe of section $\mathcal{L}_\pi$ of $E(B, B\mathcal{A}_r^*, \pi, \Psi)$ induces an isomorphism between the group of isomorphism classes of principal $B\mathcal{A}_r^*$-bundles and the group of equivalence classes of gerbes bound by $\mathcal{A}_r^*$.*

24. Corollary. *A mapping sending to the isomorphism class of a principal $B\mathcal{A}_r^*$-bundle $E(B, B\mathcal{A}_r^*, \pi, \Psi)$, $\pi : E = A\mathcal{A}_r^* \to B\mathcal{A}_r^*$, with a connection ω the equivalence class of the gerbe of section $\mathcal{L}_\pi$ of $E(B, B\mathcal{A}_r^*, \pi, \Psi)$ with the connective structure on $\mathcal{L}_\pi$ induced by ω induces an isomorphism between the group of isomorphism classes of principal $B\mathcal{A}_r^*$-bundles with connection on the group of equivalence classes of gerbes bound by $\mathcal{A}_r^*$ with connective structures.*

Proof. This follows from Proposition 19 and §22, since the case of $2 \le r \le 3$ is obtained from the complex case (see Theorems A1, A2 [13]) by additional doubling procedure of groups with doubling generators: $\mathbf{H}$ from $\mathbf{C}$ and $\mathbf{O}$ from $\mathbf{H}$, while the considered groups satisfy Conditions $4(A1, A2, C1, C2)$.

25. Sheaves, geometric bars and gerbes for wrap groups.

If G satisfies Conditions $4(A1, A2, C1, C2)$, then wrap groups $(W^ME)_{t,H}$ satisfy these Conditions $4(A1, A2, C1, C2)$ as well, since G^k satisfies them being a multiplicative subgroup of the ring $\hat{G}^k$ and $(W^{M,\{s_{0,q}:q=1,\dots,k\}}E; N, G, \mathbf{P})_{t,H}$

is the principal G^k-bundle over the commutative group $(W^{M,\{s_{0,q}:q=1,\dots,k\}}N)_{t,H}$ (see Propositions $7(1,2)$ [22]). Thus, wrap groups can be taken as the particular cases of groups for the sheaves, geometric bar and gerbes constructions (see §§1, 4, 11-13, 22, Corollary 9, Lemmas 16 [22], 17, etc.).

More concretely this can be done as follows. For a pseudo-manifold $X = X_1 \times X_2$ over $\mathcal{A}_r$, where X_1 and X_2 are H^t_p-pseudo-manifolds over $\mathcal{A}_r$, suppose that for each points $s_{0,1}, ..., s_{0,k}$ in X_1 and every neighborhood U of $\{s_{0,1}, ..., s_{0,k}\}$ in X_1 and a point $y_0 \in X_2$ and every neighborhood V of y_0 in X_2 there exist manifolds M and N such that $\{s_{0,1}, ..., s_{0,k}\} \subset M \subset U$ and $y_0 \in N \subset V$ for which a principal G-bundle $E(N, G, \pi, \Psi)$ exists with a marked group G satisfying conditions of §2 [21]. If

(1) $J(\Lambda) = \prod_{\alpha\in\Lambda} J_\alpha$

is the product of topological groups J_α, where Λ is a set, and $\Lambda_2 \subset \Lambda_1$, then there exists the natural projection group homomorphism

(2) $\hat{s}_{\Lambda_2,\Lambda_1} : J(\Lambda_1) \to J(\Lambda_2)$.

Then define a pre-sheaf F on X such that

(3) $F(U \times V) = \prod_{s_{0,1},\dots,s_{0,k}\in M\subset U; y_0\in N\subset V} (W^{M,\{s_{0,q}:q=1,\dots,k\}}E; N, G, \mathbf{P})_{t,H}$

and $s_{U_2\times V_2,U_1\times V_1} : F(U_1\times V_1) \to F(U_2\times V_2)$, since $(W^{M_2,\{s_{0,q}:q=1,\dots,k\}}E; N_2, G, \mathbf{P})_{t,H} \subset (W^{M_1,\{s_{0,q}:q=1,\dots,k\}}E; N_1, G, \mathbf{P})_{t,H}$ for $\{s_{0,q} : q = 1, ..., k\} \subset M_2 \subset M_1$ and $y_0 \in N_2 \subset N_1$ satisfying conditions of Theorem 10 [22], where $U_2 \subset U_1$ and $V_2 \subset V_1$, while open subsets of the form $U \times V$ contain the base of topology of X.

If $\mathcal{S}$ is a sheaf on X and $\mathcal{S}(U)$ satisfies Conditions $4(A1, A2, C1, C2)$ for each U open in X, then we call $\mathcal{S}$ the twisted sheaf over $\{i_0, ..., i_{2^r-1}\}$.

For $k = 1$ consider $x = \{s_0; y_0\} \in X$, but generally, consider $x = \{s_{0,1}, ..., s_{0,k}; y_0\} \in X_1^k \times X_2$ instead of $X_1 \times X_2$. Then a set $\mathcal{F}_x$ of all germs of the pre-sheaf F at a point $x \in X_1^k \times X_2$ is the inductive limit $\mathcal{F}_x = ind - \lim F(U \times V)$ taken by all open neighborhoods $U^k \times V$ of x in $X_1^k \times X_2$. Then applying the general construction of §1 gives the sheaf $\mathcal{S}_{W,X_1,X_2}$ of wrap groups. It is twisted over $\{i_0, ..., i_{2^r-1}\}$ for the group G twisted over generators $\{i_0, ..., i_{2^r-1}\}$ for $2 \le r \le 3$. This sheaf is commutative, if G is commutative.

This sheaf is obtained from the given below generalization taking a constant sheaf of the group $G = G(U)$ for each U open in X_1.

More generally, if there is a sheaf $\mathcal{G} = \mathcal{G}_{X_1}$ on X_1 of groups such that for each U open in X_1 a group $G(U)$ satisfies conditions of §2 in [21], then put

(4) $F(U\times V) = \prod_{s_{0,1},\dots,s_{0,k}\in M\subset U; y_0\in N\subset V} (W^{M,\{s_{0,q}:q=1,\dots,k\}}E; N, G(U), \mathbf{P})_{t,H}$,

where $s_{U_2,U_1} : G(U_1) \to G(U_2)$ is the restriction mapping for each $U_2 \subset U_1$ so that the parallel transport structure for $M \subset U$ is defined, $\mathcal{G}_{X_1}$, may be twisted for $2 \le r \le 3$. Therefore, due to Theorem 10 [22] and $(1, 2)$ above

there exists a restriction mapping $s_{U_2\times V_2,U_1\times V_1}: F(U_1\times V_1)\to F(U_2\times V_2)$ for each open $U_2\subset U_1$ and $V_2\subset V_1$. Then this presheaf induces a sheaf $\mathcal{S}_{W,X_1,X_2,\mathcal{G}}$ of wrap groups. If $\mathcal{G}$ is a twisted over $\{i_0,...,i_{2^r-1}\}$ for $2\le r\le 3$ sheaf, then the sheaf $\mathcal{S}_{W,X_1,X_2,\mathcal{G}}$ is twisted over $\{i_0,...,i_{2^r-1}\}$. If the sheaf $\mathcal{G}$ is commutative, then the sheaf $\mathcal{S}_{W,X_1,X_2,\mathcal{G}}$ is commutative.

26. Proposition. *If $h_j: X_j\to Y_j$ are H^t_p differentiable mappings from X_j onto Y_j, $j=1,2$, where $X=X_1\times X_2$ and $Y=Y_1\times Y_2$, X,X_1,X_2,Y,Y_1,Y_2 are H^t_p-pseudo-manifolds over $\mathcal{A}_r$, $0\le r\le 3$, $h_3:\mathcal{G}_{Y_1}\to\mathcal{G}_{X_1}$ is an H^t_p sheaf homomorphism, $t\ge[\max\{dim(X_1),dim(X_2),dim(Y_1),dim(Y_2)\}]/2+2$. Then they induce homomorphisms $(h_1,h_3)_*:\mathcal{S}_{W,Y_1,X_2,\mathcal{G}_{Y_1}}\to\mathcal{S}_{W,X_1,X_2,\mathcal{G}_{X_1}}$ and $h_{2,*}:\mathcal{S}_{W,X_1,X_2,\mathcal{G}_{X_1}}\to\mathcal{S}_{W,X_1,Y_2,\mathcal{G}_{X_1}}$ of wrap sheaves.*

Proof. If $M_2\subset U_2\subset Y_1$, then $h_1^{-1}(M_2)=:M_1\subset h_1^{-1}(U_2)=:U_1\subset X_1$ and $h_1^{-1}(U_2)=:U_1$ is open in X_1 for each U_2 open in Y_1. In view of Corollary 9 [21] and Proposition 7.1 and Theorem 10 [22] there exists a homomorphism $(h_1,h_3)_*:(W^{M_2,\{v_{0,q}:q=1,...,k_2\}}E;N,G(U_2),\mathbf{P})_{t,H}\to(W^{M_1,\{s_{0,q}:q=1,...,k_1\}}E;N,G(U_1),\mathbf{P})_{t,H}$, where $h_3: G(U_2)\to G(U_1)$ is the group homomorphism, $h_1(s_{0,q})=v_{0,a(q)}$ for each $q=1,...,k_2$, $1\le a=a(q)\le k_2$. Choose in particular $s_{0,q}$ such that $k_1=k_2=k$. Therefore, there exists the presheaf homomorphism $(h_1,h_3)_*: F_{Y_1,X_2,\mathcal{G}_{Y_1}}(U_2\times V)\to F_{X_1,X_2,\mathcal{G}_{X_1}}(U_1\times V)$ for each U_2 open in Y_1 and V open in X_2. This presheaf homomorphism induces the sheaf homomorphism.

If $f: M_1\to N_1\subset X_2$, then $h_2\circ f: M_1\to N_2$ for H^t_p pseudo-manifolds M_1 in X_1, N_1 in X_2, N_2 in Y_2. If f and h_2 are H^t_p mappings, then due to the Sobolev embedding theorem [34] for

$t\ge[\max\{dim(X_1),dim(X_2),dim(Y_1),dim(Y_2)\}]/2+2$ we have that f' exists and is continuous almost everywhere on X_1 and $h_2(f(*))$ is the H^t_p mapping (see also [6]). Then $h_{2,*}(\mathbf{P}_{\hat{\gamma},u}(x)):=\mathbf{P}_{h_2\circ\hat{\gamma},u}(x)$ implies $h_{2,*}<\mathbf{P}_{\hat{\gamma},u}>_{t,H}=<\mathbf{P}_{h_2\circ\hat{\gamma},u}>_{t,H}$ for classes of $R_{t,H}$ equivalent elements, since the group $G(U)$ and the manifold M are specified, and the same for N_1 and N_2. Therefore, there exists the induced homomorphism
$h_{2,*}:(W^{M,\{s_{0,q}:q=1,...,k\}}E;N_1,G(U),\mathbf{P})_{t,H}\to(W^{M,\{s_{0,q}:q=1,...,k\}}E;N_2,G(U),\mathbf{P})_{t,H}$, where $N_1\subset V_1\subset X_2$, $N_1=h_2^{-1}(N_2)$, $y_{0,1}\in N_1$, $h_2(y_{0,1})=y_{0,2}$, $y_{0,2}\in N_2\subset V_2\subset Y_2$. Consequently, there exists the homomorphism of pre-sheaves $h_{2,*}: F_{X_1,X_2,\mathcal{G}_{X_1}}(U\times V_1)\to F_{X_1,Y_2,\mathcal{G}_{X_1}}(U\times V_2)$ (see §25), where $V_1=h_2^{-1}(V_2)$, V_2 is open in Y_2. Thus $h_{2,*}$ induces the homomorphism of the wrap sheaves.

27. Proposition. *Let $e\to\mathcal{G}_1\to\mathcal{G}_2\to\mathcal{G}_3\to e$ be an exact sequence of sheaves on X_1. Then there exists an exact sequence $e\to\mathcal{S}_{W,X_1,X_2,\mathcal{G}_1}\to\mathcal{S}_{W,X_1,X_2,\mathcal{G}_2}\to\mathcal{S}_{W,X_1,X_2,\mathcal{G}_3}\to e$ of wrap sheaves, where e is the unit element (see §25).*

Proof. For each U open in X_1 there exists a short exact sequence of

groups $e \to G_1(U) \to G_2(U) \to G_3(U) \to e$ such that $G_3(U)$ is isomorphic with the quotient group $G_2(U)/G_1(U)$, where $G_1(U)$ is the normal closed subgroup in $G_2(U)$. In view of Theorem 10 [22] there exists the short exact sequence $e \to (W^ME; N, G_1(U), \mathbf{P})_{t,H} \to (W^ME; N, G_2(U), \mathbf{P})_{t,H} \to (W^ME; N, G_3(U), \mathbf{P})_{t,H} \to e$. Then this induces the short exact sequence of wrap presheaves $e \to F_{G_1(U)}(U) \to F_{G_2(U)}(U) \to F_{G_3(U)} \to e$ and the latter in its turn gives the short exact sequence of wrap sheaves (see also in general [3]).

28. Wrap sub-sheaf. In the construction of §25 consider a sub-presheaf corresponding to $F_N(U)$, that is, for $V = N$ with a fixed marked point $y_0 \in N$, where

(1) $F_N(U) := \prod_{s_{0,1},...,s_{0,k} \in M \subset U} (W^{M,\{s_{0,q}:q=1,...,k\}}E; N, G(U), \mathbf{P})_{t,H}$,

where $s^G_{U_2,U_1} : G(U_1) \to G(U_2)$ is the restriction mapping for each $U_2 \subset U_1$. In view of Theorem 10 [22] there exists a restriction mapping $s_{U_2,U_1} : F_N(U_1) \to F_N(U_2)$ for each open $U_2 \subset U_1$. Then this presheaf induces a sheaf $\mathcal{S}_{W,X_1,\mathcal{G}}(N)$ of wrap groups, which is the subsheaf of $\mathcal{S}_{W,X_1,X_2,\mathcal{G}}$.

29. Proposition. *Let $\eta : N_1 \to N_2$ be an $H^{t'}_p$-retraction of $H^{t'}_p$ manifolds, $N_2 \subset N_1$, $\eta|_{N_2} = id$, $y_0 \in N_2$, where $t' \ge t$, M is an H^t_p manifold, $E(N_1, G, \pi, \Psi)$ and $E(N_2, G, \pi, \Psi)$ are principal $H^{t'}_p$ bundles with a structure group G satisfying conditions of §2 [21], then there exists a sheaf homomorphism η_* from $\mathcal{S}_{W,X_1,\mathcal{G}}(N_1)$ onto $\mathcal{S}_{W,X_1,\mathcal{G}}(N_2)$.*

Proof. In view of Proposition 17 [22] there exists a group homomorphism $\eta_*(U)$ from $F_{N_1}(U)$ onto $F_{N_2}(U)$ for each U open in X_1 such that $\{s_{0,q} : q = 1, ..., k\} \subset M \subset X_1$. If $\mathcal{B}$ is a sheaf on X and $\eta\mathcal{B}(U) = \mathcal{B}(\eta^{-1}(U))$ for each U open in X, then there exists a sheaf $\eta\mathcal{B}$ which is called the image of the sheaf $\mathcal{B}$ (see [3]). On the other hand, $\eta_*(U_2) \circ s_{U_2,U_1} = s_{U_2,U_1} \circ \eta_*(U_1)$ for each open $U_2 \subset U_1$ due to Condition 25(2). Then $\mathcal{S}_{W,X_1,\mathcal{G}}(N_2)$ is the image of $\mathcal{S}_{W,X_1,\mathcal{G}}(N_1)$, that is $\eta_*\mathcal{S}_{W,X_1,\mathcal{G}}(N_1) = \mathcal{S}_{W,X_1,\mathcal{G}}(N_2)$, since there exists an H^t_p mapping $id \times \eta$ from $M \times N_1$ onto $M \times N_2$ (see §28). This gives the sheaf homomorphism (see also §3 [3]).

30. Remark. For a continuous mapping $f : X \to Y$ and a sheaf $\mathcal{B}$ on Y a inverse image $f^*\mathcal{B}$ is a sheaf on X such that $f^*\mathcal{B} = \{(x, q) \in X \times \mathcal{B} : f(x) = \pi(q)\}$ (see [3]). Particularly, if $f : X \to Y$ is an H^t_p mapping such that $f = (f_1, f_2)$, $f_1 : X_1 \to Y_1$, $f_2 : X_2 \to Y_2$, then there exists a sheaf inverse image $f^*\mathcal{S}_{W,Y_1,Y_2,\mathcal{G}_2}$, where $f_1^*\mathcal{G}_2 = \mathcal{G}_1$.

31. Corollary. *Let suppositions of Proposition 26 be satisfied, where h_j are diffeomorphisms for $j = 1, 2$ and an isomorphism for $j = 3$, then $\mathcal{S}_{W,X_1,X_2,\mathcal{G}_{X_1}}$ and $\mathcal{S}_{W,Y_1,Y_2,\mathcal{G}_{Y_1}}$ are isomorphic sheaves.*

Proof. This follows from Proposition 26 and Remark 30.

32. Proposition. *Let a sheaf $\mathcal{G}$ be an inductive limit $ind-\lim_{\alpha \in \Lambda} G_\alpha$*

of sheaves $\mathcal{G}_\alpha$, where Λ is a directed set. Then the wrap sheaf $\mathcal{S}_{W,X_1,X_2,\mathcal{G}}$ is the inductive limit $ind-\lim_{\alpha\in\Lambda}\mathcal{S}_{W,X_1,X_2,\mathcal{G}_\alpha}$.

Proof. For each U open in X_1 and all $\alpha<\beta\in\Lambda$ there exists a homomorphism $\pi^\alpha_\beta:\mathcal{G}_\alpha(U)\to\mathcal{G}_\beta(U)$. Then the sheaf $\mathcal{G}$ is defined as the sheaf generated by a pre-sheaf $U\mapsto ind-\lim_{\alpha\in\Lambda}\mathcal{G}_\alpha(U)$ (see Chapter 1 §5 [3]). Each homomorphism π^α_β generates the homomorphism of principal bundles from $E(N,G_\alpha,\pi,\Psi)$ into $E(N,G_\beta,\pi,\Psi)$. In view of Proposition 26 for each $\alpha<\beta\in\Lambda$ and every U open in X_1 there exists the group homomorphism $\pi^\alpha_{\beta,*}:\mathcal{S}_{W,X_1,X_2,\mathcal{G}_\alpha}(U)\to\mathcal{S}_{W,X_1,X_2,\mathcal{G}_\beta}(U)$ generated by π^α_β. Thus, there exists $\mathcal{S}_{W,X_1,X_2,\mathcal{G}}:=ind-\lim_{\alpha\in\Lambda}\mathcal{S}_{W,X_1,X_2,\mathcal{G}_\alpha}$.

33. Corollary. *Let $X_1=ind-\lim_{\alpha\in\Lambda}X_{1,\alpha}$ and $\mathcal{G}=ind-\lim_{\alpha\in\Lambda}\mathcal{G}_\alpha$ satisfy conditions of Proposition 26, where $\mathcal{G}_\alpha=\mathcal{G}_{X_{1,\alpha}}$ and $X_{1,\alpha}$ is an H^t_p pseudo-manifold for each α in a directed set Λ. Then $\mathcal{S}_{W,X_1,X_2,\mathcal{G}}=ind-\lim_{\alpha\in\Lambda}\mathcal{S}_{W,X_{1,\alpha},X_2,\mathcal{G}_\alpha}$.*

Proof. For an H^t_p pseudo-manifold X_1 its base of topology consists of all those subsets U in X_1 such that $U=\bigcap_{v=1}^m U_{\alpha(v)}$ for some $m\in\mathbf{N}$ and $\alpha(1),...,\alpha(m)\in\Lambda$, where $V_\alpha=\pi_\alpha^{-1}(U_{\alpha(v)})$ is open in X_α, $\pi_\alpha:X_{1,\alpha}\to X_1$ is an embedding for each α in Λ. In view of Proposition 26 for each U open in X_1 and $\alpha<\beta\in\Lambda$ there exists a group homomorphism $\pi^\alpha_{\beta,*}:\mathcal{S}_{W,X_{1,\alpha},X_2,\mathcal{G}_\alpha}(U)\to\mathcal{S}_{W,X_{1,\beta},X_2,\mathcal{G}_\beta}(U)$. Due to Proposition 32 this generates $\mathcal{S}_{W,X_1,X_2,\mathcal{G}}$ as the inductive limit of sheaves $\mathcal{S}_{W,X_{1,\alpha},X_2,\mathcal{G}_\alpha}$.

34. Theorem. *Let $X_2=X_{2,1}\times X_{2,2}$, where X_1, $X_{1,2}$ and $X_{2,2}$ are H^t_p and $H^{t'}_p$ pseudo-manifolds respectively over $\mathcal{A}_r$ as in §25. Then the restriction of the complete tensor product of wrap sheaves $\mathcal{S}_{W,X_1,X_{2,1},\mathcal{G}_1}\hat{\otimes}\mathcal{S}_{W,X_1,X_{2,2},\mathcal{G}_2}$ on $\Delta_1\times X_2$ is isomorphic with $\mathcal{S}_{W,X_1,X_2,\mathcal{G}}$, where $\mathcal{G}$ is the tensor product $\mathcal{G}:=\mathcal{G}_1\otimes\mathcal{G}_2$ of sheaves $\mathcal{G}_1$ and $\mathcal{G}_2$ on X_1, $\Delta_1:=\{(x,x):x\in X_1\}$ is the diagonal in X_1^2.*

Proof. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are sheaves on a topological space X, then $\mathcal{B}_1\otimes\mathcal{B}_2$ denotes the sheaf on X generated by the presheaf $U\mapsto\mathcal{B}_1(U)\otimes\mathcal{B}_2(U)$, where $(\mathcal{B}_1\otimes\mathcal{B}_2)_x\cong\mathcal{B}_{1,x}\otimes\mathcal{B}_{2,x}$ is the natural isomorphism of fibers. The sheaf $\mathcal{B}_1\otimes\mathcal{B}_2$ is called the tensor product of sheaves.

Consider the natural projections $\phi_1:X_2\to X_{2,1}$ and $\phi_2:X_2\to X_{2,2}$ having extensions $id\times\phi_1:X_1\times X_2\to X_1\times X_{2,1}$ and $id\times\phi_2:X_1\times X_2\to X_1\times X_{2,2}$. Therefore, there exists the sheaf $\mathcal{S}:=\mathcal{S}_{W,X_1,X_{2,1},\mathcal{G}_1}\hat{\otimes}\mathcal{S}_{W,X_1,X_{2,2},\mathcal{G}_2}:=[(id\times\phi_1)^*\mathcal{S}_{W,X_1,X_{2,1},\mathcal{G}_1}]\otimes[(id\times\phi_2)^*\mathcal{S}_{W,X_1,X_{2,2},\mathcal{G}_2}]$ which is the complete tensor product of sheaves (see in general Chapter 1 §5 [3]).

If $\gamma:M\to X_2$ is an H^t_p mapping preserving marked points, then $\gamma=(\gamma_1,\gamma_2)$, where $\gamma_j:M\to X_{2,j}$ for $j=1,2$, $\gamma(s_{0,q})=y_0$, $\gamma_j(s_{0,q})=y_{j,0}$ for each $q=1,...,k$ and $j=1,2$, $y_0=y_{1,0}\times y_{2,0}$. Then we get a lifting $\hat{\gamma}:\hat{M}\to X_2$ such that $\gamma\circ\Xi=\hat{\gamma}$ (see §§2, 3 and 6 in [21]). Therefore,

$\mathbf{P}_{\hat{\gamma},u}(\hat{s}_{0,k+q}) = \mathbf{P}_{\hat{\gamma}_1,u_1}(\hat{s}_{0,k+q}) \otimes \mathbf{P}_{\hat{\gamma}_2,u_2}(\hat{s}_{0,k+q}) \in G$ for each $q = 1, ..., k$, with $G = G_1 \otimes G_2$ being the direct product of groups for $G_1 = \mathcal{G}_1(U_1)$ and $G_2 = \mathcal{G}_2(U_2)$ for every U_j open in X_1, $j = 1, 2$, where $u \in E_{y_0}$, $u_j \in E_{j,y_{j,0}}$, $N = N_1 \times N_2$, $N_j \subset V_j \subset X_{2,j}$, $E = E(N, G, \pi, \Psi)$, $E_j = E(N_j, G_j, \pi_j, \Psi_j)$ are principal bundles, $y_0 = y_{0,1} \times y_{0,2}$, $y_{j,0} \in N_j$ are marked points, V_j is open in $X_{2,j}$ for $j = 1, 2$ (see also §25).

For classes of equivalent parallel transport structures we get $< \mathbf{P}_{\hat{\gamma},u} >_{t,H} = < \mathbf{P}_{\hat{\gamma}_1,u_1} >_{t,H} \otimes < \mathbf{P}_{\hat{\gamma}_2,u_2} >_{t,H}$, hence $F(U \times (V_1 \times V_2))$ is isomorphic with $(\phi_1)^* F(U \times V_1) \otimes (\phi_2)^* F(U \times V_2)$ for each U open in X_1 and all V_j open in $X_{2,j}$, $j = 1, 2$, since open sets of the form $V = V_1 \times V_2$ form a base of topology in X_2, where $F(U \times V_j)$ is given for the group $\mathcal{G}_j(U)$. Here $U = U_1 = U_2$ and $(\phi_1)^* F(U \times V_1) \otimes (\phi_2)^* F(U \times V_2)$ is isomorphic with the restriction of $(id \times \phi_1)^* F(U \times V_1) \otimes (id \times \phi_2)^* F(U \times V_2)$ from $U^2 \times V_1 \times V_2$ onto $\Delta(U) \times V_1 \times V_2$, where $\Delta(U)$ denotes the diagonal in U^2. Thus, $\mathcal{S}_{W,X_1,X_2,\mathcal{G}}$ is isomorphic with the restriction of the complete tensor product of sheaves $\mathcal{S}_{W,X_1,X_{2,1},\mathcal{G}_1} \hat{\otimes} \mathcal{S}_{W,X_1,X_{2,2},\mathcal{G}_2}$ on $\Delta_1 \times X_2$.

35. Twisted Alexander-Spanier cohomologies.

Let G be a group satisfying Conditions $4(A1, A2)$, which may be in particular a wrap group for $\mathcal{A}_r$ pseudo-manifolds with $2 \le r \le 3$. For an open U in X denote by $A^m(U; G)$ a group of all functions $f : U^{m+1} \to G$ with a pointwise multiplication in G as the group operation. Therefore, the functor $U \mapsto A^m(U; G)$ is a presheaf on X satisfying the condition:

$(S2)$ if $\{U_j : j\}$ is a family of open subsets of X such that $\bigcup_j U_j = U$, then for a family of elements $s_j \in A^m(U_j; G)$ such that $s_j|_{U_j \cap U_k} = s_k|_{U_j \cap U_k}$ for each j, k there exists $s \in A^m(U; G)$ such that $s|_{U_j} = s_j$ for each j. To satisfy this put $s_j = f_j : U_j^{m+1} \to G$ to be functions here and $s = f$ is their combination such that $f|_{U_j^{m+1}} = s_j$, while f on $X^{m+1} \setminus (\bigcup_j U_j^{m+1})$ is arbitrary. The property

$(S1)$ if $U = \bigcup_j U_j$, where U_j is open in X and $f, g \in A^{m+1}(U; G)$ coincide on U_j^{m+1} for each j, then $f = g$ on $\bigcup_j U_j^{m+1}$ is evident, since f, g are functions.

Recall that a family ϕ of closed subsets in X is called a family of supports, if it satisfies conditions $(SP1, SP2)$:

$(SP1)$ if B is a closed subset in C, where $C \in \phi$, then $B \in \phi$;

$(SP2)$ if $B_1, ..., B_m \in \phi$, $m \in \mathbf{N}$, then $\bigcup_{j=1}^m B_j \in \phi$.

The family ϕ of supports is called paracompactfying, if satisfies two additional conditions:

$(SP3)$ each element in ϕ is a paracompact space;

$(SP4)$ each set from ϕ has a closed neighborhood belonging to ϕ.

The union $\bigcup_{C \in \phi} C =: \mathsf{E}(\phi)$ is called a spread of ϕ. Put $\Gamma_\phi(\mathcal{S}) := \{s \in \mathcal{S}(X) : |s| \in \phi\}$ for a sheaf $\mathcal{S}$ on X, where $|s| := \{x \in X : s(x) \ne e\}$

denotes its support. Clearly $\Gamma_\phi(\mathcal{S})$ is a subgroup in $\mathcal{S}(X)$. For a presheaf A on X put $A_\phi(X) := \{s \in A(X) : |s| \in \phi\}$. For a presheaf A on X put $A_\phi(X) := \{s \in A(X) : |s| \in \phi\}$.

Let now $\mathsf{A}^m(X;G)$ be a sheaf generated by the presheaf $A^m(.;G)$. Define the differential $d : A^m(U;G) \to A^{m+1}(U;G)$ by the formula:

$df(x_0,...,x_{m+1}) = \sum_{j=0}^{m+1}(-1)^j f(x_0,...,\hat{x}_j,...,x_{m+1})$, where $f : U^{m+1} \to G$ is an arbitrary function. Then $f = \sum_{k=0}^{2^r-1} f_k i_k$, where $f_k \in \hat{G}_k$, $\{i_0,..,i_{2^r-1}\}$ are generators of $\mathcal{A}_r$, $2 \le r \le 3$. Hence d is the homomorphism of presheaves and $d^2 = 0$, since $df = \sum_{k=0}^{2^r-1}(df_k)i_k$.

Then twisted Alexander-Spanier cohomologies are defined as

${}_{AS}\mathsf{H}^m_\phi(X;G) = \mathsf{H}^m(A^*_\phi(X;G))/A^*_0(X;G))$.

36. Theorem. *Let A be a pre-sheaf on X satisfying Condition 35$(S2)$ and $\mathcal{S}$ be a sheaf generated by A, where $\mathcal{S}$ and A are twisted over $\{i_0,...,i_{2^r-1}\}$ with $1 \le r \le 3$. Then for each paracompatifying family ϕ of supports in X there exists the exact sequence*

$e \to A_0(X) \to A_\phi(X) \xrightarrow{\theta} \Gamma_\phi(\mathcal{S}) \to e$, *where $\theta : A(X) \to \mathcal{S}(X)$ is the natural mapping of the presheaf into the generated by it sheaf.*

Proof. Consider $s \in \Gamma_\phi(\mathcal{S})$ and a neighborhood U of $|s|$ such that $cl(U) \in \phi$, where $cl(U)$ denotes the closure of U in X. Since $cl(U)$ is paracompact find a locally finite covering $\{U_j : j\}$ of $cl(U)$, where each U_j is open in X and for which there exists $s_j \in A(U_j)$ such that $\theta(s_j) = s|_{U_j}$. Let $\{V_j : j\}$ be a refinement of $\{U_j : j\}$ such that $U \cap cl(V_j) \subset U_j$.

For $x \in X$ the set $J(x) := \{j : x \in cl(V_j)\}$ is finite, hence for each $x \in X$ there exists a neighborhood $W(x)$ such that $W(x) \subset U_j$ and for each $j \in J(x)$ and every $y \in W(x)$ there is the inclusion $J(y) \subset J(x)$.

For $j \in J(x)$ we get $\theta(s_j(x)) = s(x)$. Take $W(x)$ sufficiently small such that $s_j|_{W(x)} =: s_x$ does not depend on $j \in J(x)$, since $J(x)$ is finite, consequently, $s_x \in A(W(x))$.

Let $x, y \in U$, $z \in W(x) \cap W(y)$ and $j \in J(z)$, where $J(z) \subset J(x) \cup J(y)$. Then $s_x|_{W(x)\cap W(y)} = s_y|_{W(x)\cap W(y)}$. Due to Condition $(S2)$ there exists $\beta \in A(U)$ such that $\beta|_{W(x)} = s_x$ for each $x \in U$, clearly, $\theta(\beta) = s|_U$.

Take now $C \in \phi$ such that $|s| \subset Int(C)$ and $C \subset U$, where $Int(C)$ denotes the interior of C. If $x \in C \setminus Int(C)$, then $\theta(\beta)(x) = s(x) = 0$. Therefore, there exists a covering $\{Q_j\}$ of $C \setminus Int(C)$ with open in X sets Q_j such that $Q_j \subset U$ and $t|_{Q_j} = 0$ for every j.

Choose the open covering $\{Q_j\} \cup \{Int(C), X \setminus C\}$ of X and elements $e \in A(Q_j)$, $\beta|_{Int(C)} \in A(Int(C))$ and $e \in A(X \setminus C)$. Restrictions of each two elements on a common part of their domains of definition coincide. Hence due to Condition $(S2)$ such elements have a common extension $q \in A(X)$ and inevitably $\theta(q) = s$ and $|q| = |\theta(q)| = |s| \in \phi$. The sequence $e \to$

$A_0(X) \to A_\phi(X) \xrightarrow{\theta} \Gamma_\phi(\mathcal{S}) \to e$ is exact, since each subsequence $e \to A_{0,k}(X) \to A_{\phi,k}(X) \xrightarrow{\theta} \Gamma_{\phi,k}(\mathcal{S}) \to e$ is exact, where $\hat{A}_\phi = \sum_{k=0}^{2^r-1} \hat{A}_{\phi,k} i_k$ and $\hat{\Gamma}_\phi = \sum_{k=0}^{2^r-1} \hat{\Gamma}_{\phi,k} i_k$, where each $\hat{A}_{\phi,k}$ is commutative and they are pairwise isomorphic for different k, as well as $\hat{\Gamma}_{\phi,k}$ are commutative and pairwise isomorphic for different values of k, since the sheaf $\mathcal{S}$ is twisted over the group of standard generators $\{i_0, ..., i_{2^r-1}\}$ of the Cayley-Dickson algebra $\mathcal{A}_r$.

Mention, that for a pre-sheaf A satisfying Condition $(S1)$ we have $A_0(X) = e$.

37. Corollary. *Let conditions of Theorem 36 be satisfied. Then for a paracompactifying family ϕ of supports there exists the natural isomorphism:* $\mathsf{H}^m_\phi(X;G) \cong \mathsf{H}^m(\Gamma_\phi(\mathcal{S}^*(X;G)))$.

Proof. This follows immediately from Theorem 36 and §35.

38. Twisted singular cohomologies.

Let $\mathcal{B}$ be a locally finite twisted sheaf on X, that is a group $\mathcal{B}(U)$ satisfies Conditions $4(A1, A2, C1, C2)$ for each U open in X. For $U \subset X$ denote by $S^m(U;\mathcal{B})$ the group of singular m-dimensional cochains of the space U with coefficients in $\mathcal{B}$. Each element $f \in S^m(U;\mathcal{B})$ is a function posing for each m-dimensional simplex $\sigma : \Delta^m \to U$ a section $f(\sigma) \in \Gamma(\sigma^*(\mathcal{B}))$, where Δ^m is a standard m-dimensional simplex.

The pre-sheaf $S^m(.;\mathcal{B})$ satisfies Condition $(S2)$. The sheaf $\mathcal{B}$ is locally constant, then the sheaf $\sigma^*(\mathcal{B})$ is constant on Δ^m, since the simplex Δ^m is simply connected, where $m \ge 1$. Therefore, there exists a usual coboundary operator $d : S^m(U;\mathcal{B}) \to S^{m+1}(U;\mathcal{B})$.

Consider the sheaf $\mathcal{S}^m(U;\mathcal{B})$ generated by a pre-sheaf $U \mapsto S^m(U;\mathcal{B})$. Then the differential d in the pre-sheaf induces the differential in the sheaf. For a locally constant sheaf $\mathcal{B}$ singular cohomologies with coefficients in $\mathcal{B}$ and supports in the family ϕ are defined as ${}_\Delta\mathsf{H}^m_\phi(X;\mathcal{B}) = \mathsf{H}^m(S^*_\phi(X;\mathcal{B}))$. Since $\mathcal{B}$ is the twisted sheaf over $\{i_0, ..., i_{2^r-1}\}$, then $S^*_\phi(X;\mathcal{B})$ and inevitably ${}_\Delta\mathsf{H}^m_\phi(X;\mathcal{B})$ are twisted over $\{i_0, ..., i_{2^r-1}\}$.

Let $\mathcal{U} := \{U_j : j\}$ be an open covering of X and let $S^*(\mathcal{U};\mathcal{B})$ be a group of singular cochains defined on singular simplices subordinated to the covering $\mathcal{U}$. With the help of the subdivision we get, that the homomorphism $b_\mathcal{U} : S^*(X;\mathcal{B}) \to S^*(\mathcal{U};\mathcal{B})$ induces the isomorphism of cohomologies, consequently, the complex $K_\mathcal{U} = ker b_\mathcal{U}$ is acyclic. On the other hand, $S^*_0(X;\mathcal{B}) = \bigcup K^*_\mathcal{U} = ind-\lim K_\mathcal{U}$, hence $\mathsf{H}^*(S^*_0(X;\mathcal{B})) = \mathsf{H}^*(ind-\lim K^*_\mathcal{U}) = ind - \lim \mathsf{H}^*(K^*_\mathcal{U}) = e$.

Thus for a paracompatifying family of supports from the exactness of the sequence

$e \to S^*_0 \to S^*_\phi \to \Gamma_\phi(\mathcal{S}^*) \to e$

and Theorem 36 it follows the isomorphism ${}_{\triangle}\mathsf{H}^m_\phi(X;\mathcal{B}) \cong \mathsf{H}^m(\Gamma_\phi(\mathcal{S}^*(X;\mathcal{B})))$.

39. Twisted differential sheaves.

A graded sheaf is a sequence $\{\mathcal{S}^m : m \in \mathbf{Z}\}$ of sheaves, which is called a differential sheaf if there are homomorphisms

(1) $d : \mathcal{S}^m \to \mathcal{S}^{m+1}$ such that $d^2 = 0$ for each m. This sheaf may be twisted over $\{i_0, ..., i_{2^r-1}\}$, where $2 \le r \le 3$. In this case we suppose that

(2) up to an automorphisms $\theta_m : \mathcal{S}^m \to \mathcal{S}^m$ we have $\theta_{m+1} \circ d(\mathcal{S}^m_k) \subset \mathcal{S}^{m+1}_k$ for each $k = 0, ..., 2^r - 1$.

A differential sheaf $\mathcal{S}^*$ having $\mathcal{S}^m = 0$ for each $m < 0$ and supplied with the augmentation homomorphism $\varepsilon : \mathcal{B} \to \mathcal{S}^0$ is called the resolvent of the sheaf, if the sequence

$e \to \mathcal{B} \xrightarrow{\varepsilon} \mathcal{S}^0 \xrightarrow{d} \mathcal{S}^1 \xrightarrow{d} \mathcal{S}^2 \to ...$

is exact.

The notion of differential graded pre-sheaves is formulated analogously. If $\mathcal{S}^m$ is twisted, that is $\hat{\mathcal{S}}^m = \hat{\mathcal{S}}^m_0 i_0 + ... + \hat{\mathcal{S}}^m_{2^r-1} i_{2^r-1}$, where $\hat{\mathcal{S}}^m_k$ and $\hat{\mathcal{S}}^m_j$ are pairwise isomorphic and commutative for each $k \ne j$, then $Ker(d : \mathcal{S}^m \to \mathcal{S}^{m+1})$ and $Im(d : \mathcal{S}^{m-1} \to \mathcal{S}^m)$ are twisted as well, since up to isomorphisms $\theta_m : \mathcal{S}^m \to \mathcal{S}^m$ we have $\theta_{m+1} \circ d(\mathcal{S}^m_k) \subset \mathcal{S}^{m+1}_k$ for each $k = 0, ..., 2^r - 1$.

A sheaf of cohomologies (in another words a derivative sheaf) is defined as $\mathsf{H}^m(\mathcal{S}^*) = Ker(d : \mathcal{S}^m \to \mathcal{S}^{m+1})/Im(d : \mathcal{S}^{m-1} \to \mathcal{S}^m)$. If $\mathcal{S}^*$ is generated by a differential pre-sheaf S^*, then $\mathsf{H}^m(\mathcal{S}^*)$ is generated by the pre-sheaf $U \mapsto \mathsf{H}^m(S^*(U))$.

For a sheaf $\mathcal{B}$ on topological space X and an open subset $U \subset X$ denote by $\mathsf{Y}^0(U;\mathcal{B})$ a set of all mappings (may be discontinuous) $f : U \to \mathcal{B}$ such that $\pi \circ f = id$ is the identity mapping on U, where $\pi : \mathcal{B} \to X$ is the canonical projection. Thus $\mathsf{Y}^0(U;\mathcal{B}) = \prod_{x \in U} \mathcal{B}_x$ and it is the group with the pointwise group operation. Therefore, $U \mapsto \mathsf{Y}^0(U;\mathcal{B})$ is the pre-sheaf satisfying Conditions $(S1, S2)$, hence it is the sheaf which we denote by $\mathcal{Y}^0(X;\mathcal{B})$. If $\mathcal{B}$ is twisted, then $\mathcal{Y}^0(U;\mathcal{B})$ is twisted as well.

The inclusion of all continuous sections of $\mathcal{B}$ into the family of all sections not necessarily continuous induces the augmentation homomorphism $\varepsilon : \mathcal{B} \to \mathcal{Y}^0(X;\mathcal{B})$.

For a family ϕ of supports put $\mathsf{Y}^0_\phi(X;\mathcal{B}) = \Gamma_\phi(\mathcal{Y}^0(X;\mathcal{B})$. If $e \to \mathcal{B}_1 \to \mathcal{B}_2 \to \mathcal{B}_3 \to e$ is a short exact sequence of sheaves (may be twisted), then the sequence of pre-sheaves $e \to \mathsf{Y}^0(X;\mathcal{B}_1) \to \mathsf{Y}^0(X;\mathcal{B}_2) \to \mathsf{Y}^0(X;\mathcal{B}_3) \to e$ is exact. If $f \in \mathsf{Y}^0_\phi(X;\mathcal{B})$, then its support is $|f| := cl\{x : f(x) \ne e\}$. Therefore, f is an image of a section g of the sheaf $\mathcal{B}$ such that g is not necessarily continuous and $g(x) = e$ if $f(x) = e$ for $x \in X$, hence $|g| = |f| \in \phi$.

Denote by $\mathcal{Z}^1(X;\mathcal{B})$ the cokernel of the homomorphism ε such that the sequence $e \to \mathcal{B} \xrightarrow{\varepsilon} \mathcal{Y}^0(X;\mathcal{B}) \xrightarrow{\partial} \mathcal{Z}^1(X;\mathcal{B})$ is exact. Define by induction

the sheaves

$\mathcal{Y}^m(X;\mathcal{B})=\mathcal{Y}^0(X;\mathcal{Z}^m(X;\mathcal{B}))$, $\mathcal{Z}^{m+1}(X;\mathcal{B})=\mathcal{Z}^1(X;\mathcal{Z}^m(X;\mathcal{B}))$.

If $\mathcal{B}$ is twisted over $\{i_0,..,i_{2^r-1}\}$, then $\mathcal{Z}^1(X;\mathcal{B})$ is twisted as well and by induction $\mathcal{Z}^m(X;\mathcal{B})$ and $\mathcal{Y}^m(X;\mathcal{B})$ are twisted for each $m\in\mathbf{N}$. Therefore, the sequence $e\to\mathcal{Z}^m(X;\mathcal{B})\xrightarrow{\varepsilon}\mathcal{Y}^m(X;\mathcal{B})\xrightarrow{\partial}\mathcal{Z}^{m+1}(X;\mathcal{B})\to e$ is exact. Consider the composition $d=\varepsilon\circ\partial$ for $\mathcal{Y}^m(X;\mathcal{B})\xrightarrow{\partial}\mathcal{Z}^{m+1}(X;\mathcal{B})\xrightarrow{\varepsilon}\mathcal{Y}^{m+1}(X;\mathcal{B})$, then the sequence

$e\to\mathcal{B}\xrightarrow{\varepsilon}\mathcal{Y}^0(X;\mathcal{B})\xrightarrow{d}\mathcal{Y}^1(X;\mathcal{B})\xrightarrow{d}\mathcal{Y}^2(X;\mathcal{B})\to...$ is exact. Thus, $\mathcal{Y}^*(X;\mathcal{B})$ is the resolvent of the sheaf $\mathcal{B}$, which is called the canonical resolvent.

40. Proposition. *The canonical resolvent of the twisted sheaf $\mathcal{B}$ is fiberwise homotopically trivial.*

Proof. Consider the homomorphism $\mathsf{Y}^0(U;\mathcal{B})\to\mathcal{B}_x$ such that $U\ni x\mapsto f(x)\in\mathcal{B}_x$ for each $f\in\mathsf{Y}^0(U;\mathcal{B})$ and $x\in U$. The direct limit by neighborhoods of a point x induces the homomorphism $\eta_x:\mathcal{Y}^0(X;\mathcal{B})_x\to\mathcal{B}_x$, consequently, $\eta_x\circ\varepsilon:\mathcal{B}_x\to\mathcal{B}_x$ is the identity isomorphism, where $\eta_x\circ\varepsilon(z)=\eta_x(\varepsilon(z))$. Define the homomorphism $\nu_x:\mathcal{Z}^1(X;\mathcal{B})\to\mathcal{Y}^0(X;\mathcal{B})$ by the formula $\nu_x\circ\partial=1-\varepsilon\circ\eta_x$ which defines ν_x in a unique way. Therefore, there exists a fiber splitting

$\mathcal{Z}^m(X;\mathcal{B})_x\xrightarrow{\varepsilon}\mathcal{Y}^m(X;\mathcal{B})\xrightarrow{\partial}\mathcal{Z}^{m+1}(X;\mathcal{B})_x$ and

$\mathcal{Z}^m(X;\mathcal{B})_x\xleftarrow{\eta_x}\mathcal{Y}^m(X;\mathcal{B})\xleftarrow{\nu_x}\mathcal{Z}^{m+1}(X;\mathcal{B})_x$. Put $D_x:=\nu_x\circ\eta_x:\mathcal{Y}^m(X;\mathcal{B})_x\to\mathcal{Y}^{m-1}(X;\mathcal{B})_x$ for $m>0$. Therefore, $d\circ D_x+D_x\circ d=\varepsilon\circ\partial\circ\nu_x\circ\eta_x+\nu_x\circ\eta_x\circ\varepsilon\circ\partial=\varepsilon\circ\eta_x+\nu_x\circ\partial=1$ on $\mathcal{Y}^m(X;\mathcal{B})$ for $m>0$. At the same time on $\mathcal{Y}^0(X;\mathcal{B})_x$ we have $D_x\circ d=\nu_x\circ\eta_x\circ\varepsilon\circ\partial=\nu_x\circ\partial=1-\varepsilon\circ\eta_x$. This means, that $\mathcal{Y}^*(X;\mathcal{B})_x$ is homotopically fiberwise trivial resolvent.

41. Remark. The functor $\mathcal{Y}^0(X;\mathcal{B})$ is exact by $\mathcal{B}$, hence $\mathcal{Z}^1(X;\mathcal{B})$ is also the exact functor by $\mathcal{B}$. Using induction we get, that all functors $\mathcal{Y}^m(X;\mathcal{B})$ and $\mathcal{Z}^m(X;\mathcal{B})$ are exact by $\mathcal{B}$. For an arbitrary family ϕ of supports on X put $\mathsf{Y}^m_\phi(X;\mathcal{B}):=\Gamma_\phi(\mathcal{Y}^m(X;\mathcal{B}))=\mathsf{Y}^0_\phi(X;\mathcal{Z}^m(X;\mathcal{B}))$. Since the functor $\mathsf{Y}^0(X;*)$ is exact, then the functor $\mathsf{Y}^m_\phi(X;\mathcal{B})$ is exact.

42. Definition. Cohomologies in X with supports in ϕ with coefficients in $\mathcal{B}$ are defined as $\mathsf{H}^m_\phi(X;\mathcal{B}):=\mathsf{H}^m(\mathsf{Y}^*_\phi(X;\mathcal{B}))$.

42.1. Note. The sequence $e\to\Gamma_\phi(\mathcal{B})\to\Gamma_\phi(\mathcal{Y}^0(X;\mathcal{B}))\to\Gamma_\phi(\mathcal{Y}^1(X;\mathcal{B}))$ is exact, consequently, $\Gamma_\phi(\mathcal{B})\cong\mathsf{H}^0_\phi(X;\mathcal{B})$. If there is a short exact sequence of twisted sheaves $e\to\mathcal{B}_1\to\mathcal{B}_2\to\mathcal{B}_3\to e$ on X, then it implies the exact sequence of cochain complexes

$e\to\mathsf{Y}^*_\phi(X;\mathcal{B}_1)\to\mathsf{Y}^*_\phi(X;\mathcal{B}_2)\to\mathsf{Y}^*_\phi(X;\mathcal{B}_3)\to e$, that in its turn induces the long exact sequence

$...\to\mathsf{H}^m_\phi(X;\mathcal{B}_1)\to\mathsf{H}^m_\phi(X;\mathcal{B}_2)\to\mathsf{H}^m_\phi(X;\mathcal{B}_3)\xrightarrow{\delta}\mathsf{H}^{m+1}_\phi(X;\mathcal{B}_1)\to....$

43. Definition. Let G be a topological group satisfying conditions

$4(A1, A2, C1, C2)$ such that G is a multiplicative group of the ring $\hat{G}$, where $1 \le r \le 2$. Then define the smashed product G^s such that it is a multiplicative group of the ring $\hat{G}^s := \hat{G} \otimes_l \hat{G}$, where $l = i_{2^r}$ denotes the doubling generator, the multiplication in $\hat{G} \otimes_l \hat{G}$ is

(1) $(a + bl)(c + vl) = (ac - v^* b) + (va + bc^*)l$ for each $a, b, c, v \in \hat{G}$, where $v^* = conj(v)$.

A smashed product $M_1 \otimes_l M_2$ of manifolds M_1, M_2 over $\mathcal{A}_r$ with $dim(M_1) = dim(M_2)$ is defined to be an $\mathcal{A}_{r+1}$ manifold with local coordinates $z = (x, yl)$, where x in M_1 and y in M_2 are local coordinates.

44. Theorem. *There exists smashed products $\mathcal{S}^s := \mathcal{S}_1 \otimes_l \mathcal{S}_2$ on $X = X_1 = X_2$ and $\hat{\mathcal{S}}^s := \mathcal{S}_1 \hat{\otimes}_l \mathcal{S}_2$ on $X = X_1 \times X_2$ over $\{i_0, ..., i_{2^{r+1}-1}\}$ of isomorphic twisted sheaves $\mathcal{S}_1$ on X_1 and $\mathcal{S}_2$ on X_2 over $\{i_0, ..., i_{2^r-1}\}$ with $X_1 = X_2$, in particular of wrap sheaves, where $1 \le r \le 2$, $l = i_{2^r}$.*

Proof. If $\mathcal{S}_j$ is a sheaf on a topological space X_j twisted over $\{i_0, ..., i_{2^r-1}\}$, then $\hat{\mathcal{S}}_j = \hat{\mathcal{S}}_{0,j} i_0 \oplus ... \oplus \hat{\mathcal{S}}_{2^r-1,j} i_{2^r-1}$, where $\hat{\mathcal{S}}_{k,j}(U) = \mathcal{S}_{k,j}(U) \cup \{0\}$ are commutative rings for each U open in X_j, $\hat{\mathcal{S}}_{k,j}$ are sheaves on X_j pairwise isomorphic for different values of k. Then for $X = X_1 = X_2$ take $\mathcal{S}^s_x := (\mathcal{S}_1)_x \otimes_l (\mathcal{S}_2)_x$ for each $x \in X$ in accordance with Definition 43, that defines the twisted sheaf $\mathcal{S}$ on X over $\{i_0, ..., i_{2^{r+1}-1}\}$ due to Proposition 19 [22]. This sheaf $\mathcal{S}$ is the smashed tensor product of sheaves.

If $X = X_1 \times X_2$, then take $\hat{\mathcal{S}}^s := \mathcal{S}_1 \hat{\otimes}_l \mathcal{S}_2 = (\pi_1^* \mathcal{S}_1) \otimes_l (\pi_2^* \mathcal{S}_2)$, which is the smashed complete tensor product of sheaves, where $\pi_1 : X \to X_1$ and $\pi_2 : X \to X_2$ are projections.

45. Corollary. *Let $X_2 = X_{2,1} \otimes_l X_{2,2}$ be the smashed product, where X_1 and X_2 are H^t_p and $H^{t'}_p$ pseudo-manifolds respectively over $\mathcal{A}_{r+1}$, $1 \le r \le 2$. Then the restriction the smashed complete tensor product of wrap sheaves $\mathcal{S}_{W,X_1,X_{2,1},\mathcal{G}} \hat{\otimes}_l \mathcal{S}_{W,X_1,X_{2,2},\mathcal{G}}$ on $\Delta_1 \times X_2$ is isomorphic with $\mathcal{S}_{W,X_1,X_2,\mathcal{G}^s}$, where $\mathcal{G}^s$ is the smashed tensor product $\mathcal{G}^s := \mathcal{G} \otimes_l \mathcal{G}$ twisted over $\{i_0, ..., i_{2^{r+1}-1}\}$ of a sheaf $\mathcal{G}$ twisted over $\{i_0, ..., i_{2^r-1}\}$ on X_1, $\Delta_1 := \{(x, x) : x \in X_1\}$ is the diagonal in X_1^2.*

Proof. The smashed product of manifolds was described in details in the proof of Theorem 20 [22]. Consider an $\mathcal{A}_r$ shadow of X_1 that exists, since $\mathcal{A}_{r+1} = \mathcal{A}_r \oplus \mathcal{A}_r l$, where $l = i_{2^r}$. For each U open in X_1 there exists a group $\mathcal{G}(U)$, hence $\mathcal{G}(U) \otimes_l \mathcal{G}(U)$ is defined due to Proposition 19 [22], that gives the sheaf $\mathcal{G}^s$ on X_1. Then wrap sheaves $\mathcal{S}_{W,X_1,X_{2,b},\mathcal{G}}$ over $\mathcal{A}_r$ are defined, where $b = 1, 2$. Thus the statement of this corollary follows from Proposition 19 [22] and Theorem 44, modifying the proof of §34 for the smashed complete tensor product instead of complete tensor product so that $\mathbf{P}_{\hat{\gamma},u}(\hat{s}_{0,k+q}) = \mathbf{P}_{\hat{\gamma}_1,u_1}(\hat{s}_{0,k+q}) \otimes_l \mathbf{P}_{\hat{\gamma}_2,u_2}(\hat{s}_{0,k+q}) \in G^s$ with $E = E(N, G^s, \pi, \Psi)$, where $G = \mathcal{G}(U)$, $U = U_1 = U_2$, consequently, $< \mathbf{P}_{\hat{\gamma},u} >_{t,h} = < \mathbf{P}_{\hat{\gamma}_1,u_1} >_{t,H}$

$\otimes_l < \mathbf{P}_{\hat{\gamma}_2,u_2} >_{t,H}$.

46. Corollary. *Let $X_1 = X_{1,1} \otimes_l X_{1,2}$ and $X_2 = X_{2,1} \otimes_l X_{2,2}$ are smashed products, where X_1, and X_2 are H^t_p and $H^{t'}_p$ pseudo-manifolds respectively over $\mathcal{A}_{r+1}$, $1 \le r \le 2$. Then the wrap sheaf $\mathcal{S}_{W,X_1,X_2,\mathcal{G}^s}$ is twisted over $\{i_0,...,i_{2^{r+1}-1}\}$ and is isomorphic with the smashed complete tensor product of twice iterated wrap sheaves*

$\mathcal{S}_{W,X_{1,2},X_{2,1},\mathcal{S}_{W,X_{1,1},X_{2,1},\mathcal{G}}} \hat{\otimes}_l \mathcal{S}_{W,X_{1,2},X_{2,2},\mathcal{S}_{W,X_{1,1},X_{2,2},\mathcal{G}}}$,

where $\mathcal{G}^s$ is the smashed tensor product $\mathcal{G}^s := \mathcal{G} \otimes_l \mathcal{G}$ of a twisted sheaf $\mathcal{G}$ over $\{i_0,...,i_{2^r-1}\}$ on X_1.

Proof. Consider projections $\pi_{b,j} : X_b \to X_{b,j}$, where $j, b = 1, 2$. Each $\mathcal{A}_{r+1}$ manifold has the shadow which is the $\mathcal{A}_r$ manifold, since $\mathcal{A}_{r+1} = \mathcal{A}_r \oplus \mathcal{A}_r l$. If U is open in $X_{1,j}$, then $\pi^{-1}_{1,j}(U)$ is open in X_1 and there exists a group $\mathcal{G}(\pi^{-1}_{1,j}(U))$, where $j = 1, 2$.

Hence there exist the projection sheaves $\mathcal{G}_j = \pi^{-1}_{1,j}\mathcal{G}$ on $X_{1,j}$ induced by $\mathcal{G}$ such that $\mathcal{G}_j(U) := \mathcal{G}(\pi^{-1}_{1,j}(U))$. Denote $\mathcal{G}_j$ on $X_{1,j}$ also by $\mathcal{G}$, since $\mathcal{G}_j$ is obtained from $\mathcal{G}$ by taking the specific subfamily of open subsets. For U_1 open in $X_{1,1}$ and U_2 open in $X_{1,2}$ take $U = U_1 \times U_2$ open in X_1. The family of all such subsets gives the base of the topology in X_1.

In accordance with Definition 43 there exists $\hat{\mathcal{G}}(U) \otimes_l \hat{\mathcal{G}}(U) =: \hat{\mathcal{G}}^s(U)$, that induces $\mathcal{G}^s$ on X_1 such that $\hat{\mathcal{G}}^s_x = \hat{\mathcal{G}}_x \otimes_l \hat{\mathcal{G}}_x$ for each $x \in X_1$. Therefore, every element $q + vl$ is in $\hat{\mathcal{G}}^s(U)$ for each $q, v \in \hat{\mathcal{G}}(U)$. Thus the statement of this corollary follows from §25, Theorems 20 [22] and 44.

47. Consider now the iterated wrap sheaf $\mathcal{S}_{W,X_1,X_2,\mathcal{G};b}$ of iterated wrap groups $(W^M E)_{b,\infty,H}$ with $b \in \mathbf{N}$ instead of wrap groups for $b = 1$ such that for its presheaf

(1) $F_b(U \times V) = \prod_{s_{0,1},...,s_{0,k} \in M \subset U; y_0 \in N \subset V} (W^{M,\{s_{0,q}:q=1,...,k\}}E; N, G(U), \mathbf{P})_{b;\infty,H}$,

where $s_{U_2,U_1} : G(U_1) \to G(U_2)$ is the restriction mapping for each $U_2 \subset U_1$ so that the parallel transport structure for $M \subset U$ is defined, where $\mathcal{G}$ is the sheaf on X_1, $G(U) = \mathcal{G}(U)$, pseudo-manifolds X_1 and X_2 and the sheaf $\mathcal{G}$ are of class H^∞_p (see also §25).

Corollary. *There exists a homomorphism of of iterated wrap sheaves $\theta : \mathcal{S}_{W,X_1,X_2,\mathcal{G};a} \otimes \mathcal{S}_{W,X_1,X_2,\mathcal{G};b} \to \mathcal{S}_{W,X_1,X_2,\mathcal{G};a+b}$ for each $a, b \in \mathbf{N}$. Moreover, if $\mathcal{G}$ is either associative or alternative, then θ is either associative or alternative.*

Proof. For pre-sheaves the mapping

(2) $\theta : F_a(U \times V) \otimes F_b(U \times V) \to F_{a+b}(U \times V)$

is induced by Formula 47(1) and due to Theorem 21 [22]. Then θ has the extension on the sheaf of iterated wrap groups, since $(\mathcal{S}_{W,X_1,X_2,\mathcal{G};a})_z = ind - \lim F_a(U \times V)$, where the direct limit is taken by open subsets $U \times V$ for a point $z = x \times y \in X^k_1 \times X_2$, $x \in X^k_1$, $y \in X_2$, such that $x \subset U$, $y \in V$, U is open in X_1, V is open in X_2.

The inductive limit topology in $(\mathcal{S}_{W,X_1,X_2,\mathcal{G};a})_z$ is the finest topology rel-

ative to which each embedding $F_a(U \times V) \hookrightarrow (\mathcal{S}_{W,X_1,X_2,\mathcal{G};a})_z$ is continuous. If $f \in (\mathcal{S}_{W,X_1,X_2,\mathcal{G};a})_z$ and $g \in (\mathcal{S}_{W,X_1,X_2,\mathcal{G};b})_z$, then there exist open $U_1 \times V_1$ and $U_2 \times V_2$ such that $f \in F_a(U_1 \times V_1)$ and $g \in F_b(U_2 \times V_2)$, consequently, $f \in F_a(U \times V)$ and $g \in F_b(U \times V)$, where $U = U_1 \cup U_2$ and $V = V_1 \cup V_2$, hence $\theta(f, g) \in F_{a+b}(U \times V)$. From (2) and the definition of the inductive limit topology it follows, that θ is continuous, since on iterated wrap groups θ is H_p^∞ differentiable.

Moreover, in accordance with Theorem 21 [22] θ is either associative or alternative if $\mathcal{G}$ is associative or alternative.

48. Note. Let ϕ be a family of supports in X and $\mathcal{B}$ be a sheaf on X, where $\mathcal{B}$ may be twisted. A sheaf $\mathcal{B}$ is called ϕ-acyclic, if $\mathsf{H}^b_\phi(X; \mathcal{B}) = 0$ for each $b > 0$.

Let $\mathcal{L}^*$ be a resolvent of $\mathcal{B}$. Put $\mathcal{Z}^b := Ker(\mathcal{L}^b \to \mathcal{L}^{b+1}) = Im(\mathcal{L}^{b-1} \to \mathcal{L}^b)$, where $\mathcal{Z}^0 = \mathcal{B}$. An exact sequence

(1) $e \to \mathcal{Z}^{b-1} \to \mathcal{L}^{b-1} \to \mathcal{Z}^b \to e$

induces an exact sequence

(2) $e \to \Gamma_\phi(\mathcal{Z}^{b-1}) \to \Gamma_\phi(\mathcal{L}^{b-1}) \to \Gamma_\phi(\mathcal{Z}^b) \to \mathsf{H}^1_\phi(X; \mathcal{Z}^{b-1})$.

Therefore, there exists the monomorphism

(3) $\mathsf{H}^b(\Gamma_\phi(\mathcal{L}^*)) = \Gamma_\phi(\mathcal{Z}^b)/Im(\Gamma_\phi(\mathcal{L}^{b-1} \to \Gamma_\phi(\mathcal{Z}^b)) \to \mathsf{H}^1_\phi(X; \mathcal{Z}^{b-1})$.

Moreover, the sequence $e \to \mathcal{Z}^{b-v} \to \mathcal{L}^{b-v} \to \mathcal{Z}^{b-v+1} \to e$ induces the homomorphism:

(4) $\mathsf{H}^{b-1}_\phi(X; \mathcal{Z}^{b-v+1}) \to \mathsf{H}^v_\phi(X; \mathcal{Z}^{b-v})$.

Define κ as the composition

(5) $\mathsf{H}^b(\Gamma_\phi(\mathcal{L}^*)) \to \mathsf{H}^1_\phi(X; \mathcal{Z}^{b-1}) \to \mathsf{H}^2_\phi(X; \mathcal{Z}^{b-2}) \to ... \to \mathsf{H}^b_\phi(X; \mathcal{Z}^0)$.

If all sheaves $\mathcal{L}^b$ are ϕ-acyclic, then $(3, 4)$ are isomorphisms. We call κ natural, if from the commutativity of the diagram:

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & \mathcal{L}^* \\ \downarrow f & & \downarrow g \\ \mathcal{E} & \longrightarrow & \mathcal{M}^* \end{array}$$

where g is a homomorphism of resolvents the commutativity of the diagram

$$\begin{array}{ccc} \mathsf{H}^b(\Gamma_\phi(\mathcal{L}^*) & \xrightarrow{\kappa} & \mathsf{H}^b_\phi(X; \mathcal{B}) \\ \downarrow g^* & & \downarrow f^* \\ \mathsf{H}^b(\Gamma_\phi(\mathcal{M}^*) & \xrightarrow{\kappa} & \mathsf{H}^b_\phi(X; \mathcal{E}) \end{array}$$

follows. Thus we get the statement.

48.1. Theorem. *If $\mathcal{L}^*$ is the resolvent of the sheaf $\mathcal{B}$, consisting of ϕ-acyclic sheaves, then for each $b \in \mathbf{N}$ the natural mapping*

$\kappa : \mathsf{H}^b(\Gamma_\phi(\mathcal{L}^*) \to \mathsf{H}^b_\phi(X; \mathcal{B})$ is the isomorphism.

In view of the latter theorem if $g: \mathcal{L}^* \to \mathcal{M}^*$ is the homomorphism of two resolvents of the sheaf $\mathcal{B}$ consisting of ϕ-acyclic sheaves, then the induced mapping $\mathsf{H}^b(\Gamma_\phi(\mathcal{L}^*)) \to \mathsf{H}^b(\Gamma_\phi(\mathcal{M}^*))$ is an isomorphism.

48.2. Corollary. *If $e \to \mathcal{L}^0 \to \mathcal{L}^1 \to \mathcal{L}^2 \to ...$ is an exact sequence of ϕ-acyclic sheaves, then the corresponding sequence $e \to \Gamma_\phi(\mathcal{L}^0) \to \Gamma_\phi(\mathcal{L}^1) \to \Gamma_\phi(\mathcal{L}^2) \to ...$ is exact.*

Proof. In view of Theorem 48.1 $\mathsf{H}^b(\Gamma_\phi(\mathcal{L}^*)) = \mathsf{H}^b_\phi(X; e)$. On the other hand, $\mathsf{Y}^n_\phi(X; e) = e$, since $\mathcal{Y}^0(X; e) = e$ and hence $\mathcal{Y}^n(X; e) = e$ for all n, consequently, $\mathsf{H}^b(\Gamma_\phi(\mathcal{L}^*)) = e$ for each b.

49. Differential forms and twisted cohomologies over octonions. A bar resolution exists for any sheaf or a complex of sheaves. Consider differential forms on N. In local coordinates write a differential k-form as

(1) $w = \sum_J f_J(z) dx_{b_1,j_1} \wedge dx_{b_2,j_2} \wedge ... \wedge dx_{b_k,j_k}$,

where $f_J: N \to \mathcal{A}_r$, $z = (z_1, z_2, ...)$ are local coordinates in N, $z_b = x_{b,0}i_0 + x_{b,1}i_1 + ... + x_{b,2^r-1}i_{2^r-1}$, where $z_b \in \mathcal{A}_r$, $x_{b,j} \in \mathbf{R}$ for each b and every $j = 0, 1, ..., 2^r - 1$, $J = (b_1, j_1; b_2, j_2; ...; b_k, j_k)$. For the sheaf $\mathcal{S}^k_{N,\mathcal{A}_r}$ of germs of $\mathcal{A}_r$ valued k-forms on N has a bar resolution:

(2) $0 \to \mathcal{S}^k_{N,\mathcal{A}_r} \xrightarrow{\sigma} \mathcal{S}^k_{N,A\mathcal{A}_r} \xrightarrow{\sigma} \mathcal{S}^k_{N,AB\mathcal{A}_r} \xrightarrow{\sigma} ...,$

where $\mathcal{S}^k_{N,AB^m\mathcal{A}_r}$ denotes the sheaf of germs of $AB^m\mathcal{A}_r$ valued k-forms on N.

Denote by $\mathbf{Z}(q, \mathcal{C}_r)$ the group analogous to $\mathbf{Z}(\mathcal{C}_r)$ with $u \in \mathcal{C}_r$ replaced on u^q, where u^q is considered as equivalent with $(-u)^q$, $q \in \mathbf{N}$. Therefore, the exponential sequence

(3) $0 \to \mathbf{Z}(\mathcal{C}_r)_N \xrightarrow{\eta} \mathsf{C}^\infty(N, \mathcal{A}_r) \xrightarrow{\exp} \mathsf{C}^\infty(N, \mathcal{A}^*_r) \to 0$

can be considered as a quasi-isomorphism:

$$\begin{array}{ccc} \mathbf{Z}(\mathcal{C}_r)_N & \xrightarrow{\eta} & \mathsf{C}^\infty(N, \mathcal{A}_r) \\ \downarrow & & \downarrow \exp \\ 0 & \longrightarrow & \mathsf{C}^\infty(N, \mathcal{A}^*_r) \end{array}$$

between the complex $\mathbf{Z}(\mathcal{C}_r)^\infty_D : \mathbf{Z}(\mathcal{C}_r)_N \to \mathsf{C}^\infty(N, \mathcal{A}_r)$ and the sheaf $\mathsf{C}^\infty(N, \mathcal{A}^*_r)$ of germs of C^∞ functions from N into $\mathcal{A}^*_r$ placed in degree one, that is $\mathsf{C}^\infty(N, \mathcal{A}^*_r)[-1]$, where $\eta(z) = 2\pi z$ for each z and $\exp(0) = 1$ (see also §19), $\mathcal{A}_r$ is considered as the additive group $(\mathcal{A}_r, +)$, while $\mathcal{A}^*_r$ is the multiplicative group $(\mathcal{A}^*_r, \times)$. More generally this gives the quasi-isomorphism:

(4) $\mathbf{Z}(1, \mathcal{C}_r)_N \longrightarrow \mathsf{C}^\infty(N, \mathcal{A}_r) \xrightarrow{d} \mathcal{S}^1_{N,\mathcal{A}_r} \xrightarrow{d} ... \xrightarrow{d} \mathcal{S}^{q-1}_{N,\mathcal{A}_r}$ and

$0 \longrightarrow \mathsf{C}^\infty(N, \mathcal{A}^*_r) \xrightarrow{dLn} \mathcal{S}^1_{N,\mathcal{A}_r} \xrightarrow{d} ... \xrightarrow{d} \mathcal{S}^{q-1}_{N,\mathcal{A}_r}$

with vertical homomorphisms $\mathbf{Z}(1, \mathcal{C}_r)_N \to 0$, $\mathsf{C}^\infty(N, \mathcal{A}_r) \xrightarrow{e} \mathsf{C}^\infty(N, \mathcal{A}^*_r)$, $\mathcal{S}^1_{N,\mathcal{A}_r} \xrightarrow{id} \mathcal{S}^1_{N,\mathcal{A}_r}, ..., \mathcal{S}^{q-1}_{N,\mathcal{A}_r} \xrightarrow{id} \mathcal{S}^{q-1}_{N,\mathcal{A}_r}$ for $2 \le q \in \mathbf{N}$, where $e(f) := \exp(f)$ between a degree q smooth twisted complex

(5) $\mathbf{Z}(\mathcal{C}_r)^\infty_D : \mathbf{Z}(\mathcal{C}_r)_N \to \mathsf{C}^\infty(N, \mathcal{A}_r) \xrightarrow{d} \mathcal{S}^1_{N,\mathcal{A}_r} \xrightarrow{d} ... \xrightarrow{d} \mathcal{S}^{q-1}_{N,\mathcal{A}_r}$

and the complex $\mathcal{S}^{<q}(N, \mathcal{A}_r)(dLn)[-1]$, where

(6) $\mathcal{S}^{<q}(N,\mathcal{A}_r)(dLn): \mathsf{C}^\infty(N,\mathcal{A}_r^*)\xrightarrow{dLn}\mathcal{S}^1_{N,\mathcal{A}_r}\xrightarrow{d}...\xrightarrow{d}\mathcal{S}^{q-1}_{N,\mathcal{A}_r}$.

The hypercohomology ${}_h\mathsf{H}^q(N,\mathbf{Z}(\mathcal{C}_r)^\infty_D)$ of $\mathbf{Z}(\mathcal{C}_r)^\infty_D$ is a twisted non-commutative for $r=2$ and non-associative analog for $r=3$ of smooth Deligne cohomology of N, since $\mathcal{A}_2=\mathbf{H}=\mathbf{C}\oplus i_2\mathbf{C}$ and $\mathcal{A}_3=\mathbf{O}=\mathbf{C}\oplus i_2\mathbf{C}\oplus i_4\mathbf{C}\oplus i_6\mathbf{C}$ are quaternion and octonion algebras over $\mathbf{R}$ with the corresponding twisted structures causing twisted structures of AG and BG as above. Thus hypercohomolgies have induced twisted structures. We have that $\mathcal{S}^{<q}(N,\mathcal{A}_r)(dLn))$ is a trancation of the acyclic resolution (6) of the constant sheaf $(\mathcal{A}_r^*)_N$. Therefore, the quasi-isomorphism (5) implies

(7) ${}_h\mathsf{H}^b(N,\mathbf{Z}(\mathcal{C}_r)^\infty_D)\cong {}_h\mathsf{H}^{b-1}(\mathcal{S}^{<q}(N,\mathcal{A}_r)(dLn))$ for each b and q.

For the covering dimension $b=dimN$ (see [10]) there are the isomorphisms:

(8) ${}_h\mathsf{H}^b(\mathcal{S}^{<b+1}(N,\mathcal{A}_r)(dLn))\xrightarrow{e_N}{}_h\mathsf{H}^b(N,\mathcal{A}_r^*)\xrightarrow{t^b_N}\mathcal{A}_r^*$, the composition of which is the isomorphism:

(9) $\mathsf{T}^b_N: {}_h\mathsf{H}^b(\mathcal{S}^{<b+1}(N,\mathcal{A}_r)(dLn))\longrightarrow\mathcal{A}_r^*$.

There is useful the short exact sequence of complexes of sheaves:

(10) $0\to(\mathcal{A}_r^*)_N\to\mathsf{C}^\infty(N,\mathcal{A}_r^*)\xrightarrow{dLn}\mathcal{S}^1(N,\mathcal{A}_r)\xrightarrow{d}...\xrightarrow{d}\mathcal{S}^q(N,\mathcal{A}_r)\xrightarrow{d}\mathcal{S}^{q+1,cl}(N,\mathcal{A}_r)\to 0$,

where $\mathcal{S}^{q+1,cl}(N,\mathcal{A}_r)$ denotes the sheaf of germs of closed $\mathcal{A}_r$ valued $q+1$ forms on N.

50. Remark. Consider an open covering $\mathcal{V}:=\{V_j: j\in J\}$ of a H^∞ manifold N, denote by $\mathcal{T}(E):=\{g_j: g_j\in\Gamma(V_j,E), j\in J\}$ a family of local trivializations of E, where J is a set. If $V_k\cap V_j\ne\emptyset$, then the quotient $g_{k,j}:=g_k(1/g_j)$ is an H^∞ smooth $\mathcal{A}_r^*$-valued function on $V_k\cap V_j$, where $1\le r\le 3$. If $1\le r\le 2$, then $\mathcal{A}_r$ is associative and $g_{k,j}g_{j,l}=g_{k,l}$ on $V_k\cap V_j\cap V_l$, when the latter set is not empty.

For $r=3$ the octonion algebra $\mathbf{O}$ is only alternative and generally $g_{k,j}g_{j,l}$ may be different from $g_{k,l}$. Already for quaternions and moreover for octonions $Ln(xy)$ generally may be different from $Ln(x)+Ln(y)$ for $x,y\in\mathcal{A}_r$ with $2\le r\le 3$ because of non-commutativity.

In view of Proposition 3.2 [27, 28] for each $x,y\in\mathcal{A}_r$ there exists $z\in\mathcal{A}_r$ such that

(1) $e^xe^y=e^z=e^{a+b}e^{K(M,N)}$, where $a=Re(x)$, $b=Re(y)$, $M=x-Re(x)=:Im(x)$, $N=Im(y)$, $K=Im(z)$. As usually we denote by ln the natural logarithmic function in the commutative case $0\le r\le 1$, while Ln denotes the natural logarithmic function over $\mathcal{A}_r$ when $2\le r$ (see Section 3 in [27, 28] and [32]). The logarithmic function is defined on $\mathcal{A}_r\setminus\{0\}$ for non-zero Cayley-Dickson numbers and has a non-commutative analog of the Riemann surface so that exp and Ln are $\mathcal{A}_r$ holomorphic. For each Cayley-Dickson number v in the multiplicative group $\mathcal{A}_r^*=\mathcal{A}_r\setminus\{0\}$ there exists

$x \in \mathcal{A}_r$ such that $e^x = v$. Then

(2) $Ln(e^x e^y) = Ln(e^z) = a + b + K(M,N)$, where

(3) $K(M,N) - M - N =: P(M,N)$ may be non-zero. Express the real part as

(4) $Re(z) = (z + (2^r - 2)^{-1}\{-z + \sum_{j=1}^{2^r-1} i_j(zi_j^*)\})/2$, then

(5) $Im(z) = z - Re(z) = (z - (2^r - 2)^{-1}\{-z + \sum_{j=1}^{2^r-1} i_j(zi_j^*)\})/2$

and fix these z-representations with which $M = M(x)$, $N = N(y)$ and $P(M,N)$ are locally analytic functions by x and y. Put

(6) $Ln(f_k) = w_k$ and

(7) $Ln(g_{k,j}) = w_k - w_j + \nu_{k,j}$ and

(8) $Ln(g_{k,l}) = Ln(g_{k,j}) + Ln(g_{j,l}) + \eta_{k,j,l}$,

so that w_k and $\nu_{k,j}$ and $\eta_{k,j,l}$ are H^∞ differential 1-forms. Then from $(6-8)$ it follows, that

(9) $w_k - w_l + \nu_{k,l} = w_k - w_j + \nu_{k,j} + w_j - w_l + \nu_{j,l} + \eta_{k,j,k}$ and hence

(10) $\eta_{k,j,l} = \nu_{k,l} - \nu_{k,j} - \nu_{j,l}$.

Generally $\eta_{k,j,l}$ may be non-zero because of non-commutativity or non-associativity.

In view of the alternativity of the octonion algebra $\mathbf{O}$ the identities $e^M e^N = e^K$, $e^M = e^K e^{-N}$, $e^N = e^{-M} e^K$ and $e^{-K} = e^{-N} e^{-M}$ are equivalent, that leads to the identities:

(11) $M = K(K(M,N),-N)$, $N = K(-M, K(M,N))$, $K(M,N) = -K(-N,-M)$, where M, N, K are purely imaginary octonions, moreover, $K(M,0) = M$, $K(0,N) = N$, since $e^0 = 1$.

Let $E(N, \mathcal{A}_r^*, \pi, \Psi)$ be an H^∞ principal $\mathcal{A}_r^*$-bundle with transition functions $\{g_{k,j} : V_k \cap V_j \to \mathcal{A}_r^* : k, j\}$ and consider a family $\{w_k, \nu_{k,j}, \eta_{k,j,l} : k, j, l\}$ of 1-forms related by Equations $(6-8)$ so that $w_j \in \Gamma(V_j, \mathcal{S}^1_{N,\mathcal{A}_r})$, $\nu_{k,j} \in \Gamma(V_k \cap V_j, \mathcal{S}^1_{N,\mathcal{A}_r})$ for $V_k \cap V_j \ne \emptyset$, $\eta_{k,j,l} \in \Gamma(V_k \cap V_j \cap V_l, \mathcal{S}^1_{N,\mathcal{A}_r})$ for $V_k \cap V_j \cap V_l \ne \emptyset$, where $k, j, l \in J$.

Consider a C^∞ partition of unity $\{f_j : j \in J\}$ subordinated to the covering $\mathcal{V}$. Then

(12) $-w(x) = |f_{j_0}, f_{j_1}, ..., f_{j_n}, -w_{j_0}(x), -w_{j_1}(x), ..., -w_{j_n}(x)|$ and

(13) $-\nu(x) = |f_{j_0} f_{k_0}, f_{j_1} f_{k_1}, ..., f_{j_n} f_{k_n}, -\nu_{j_0,k_0}(x), -\nu_{j_1,k_1}(x), ..., -\nu_{j_n,k_n}(x)|$

and

(14) $-\eta(x) = |f_{j_0} f_{k_0} f_{l_0}, f_{j_1} f_{k_1} f_{l_1}, ..., f_{j_n} f_{k_n} f_{l_n}, -\eta_{j_0,k_0,l_0}(x), -\nu_{j_1,k_1,l_1}(x), ..., -\nu_{j_n,k_n,l_n}(x)|$,

where $w_j(x)$ and $\nu_{j,k}(x)$ and $\eta_{j,k,l}(x)$ denote the restriction of w_j and $\nu_{j,k}$ and $\eta_{j,k,l}$ to $T_x N$ so that $w_j(x)$ and $\nu_{j,k}(x)$ and $\eta_{k,j,l}(x)$ are $A\mathcal{A}_r$-valued 1-forms on N,

(15) $\pi_*(-w(x)) = |f_{j_0}, f_{j_1}(x), ..., f_{j_n}(x); [w_{j_0}(x) - w_{j_1}(x) + \nu_{j_0,j_1}(x)| ... |w_{j_{n-1}}(x) - w_{j_n}(x) + \nu_{j_{n-1},j_n}]|$,

where $\pi : E\mathcal{A}_r \to B\mathcal{A}_r$ is the standard projection.

The principal G-bundle $E(N, G, \pi, \Psi)$ is a pull-back of the universal bun-

dle $AG \to BG$ by a classifying mapping $g_{E(N,G,\pi,\Psi)} : N \to BG$. In terms of transition functions

(16) $g_{E(N,\mathcal{A}_r^*,\pi,\Psi)} = |f_{j_0}(x), f_{j_1}(x), ..., f_{j_n}(x); [g_{j_0,j_1}(x)|g_{j_1,j_2}(x)|...|g_{j_{n-1},j_n}(x)]|$.
Therefore,

(17) $\pi_*(w) + dLn(g_{E(N,\mathcal{A}_r^*,\pi,\Psi)}) = 0$,
where for any differentiable function $g : U \to B\mathcal{A}_r^*$ we have

$g(x) = |f_0(x), f_1(x), ..., f_n(x); [g_1(x)|...|g_n(x)]|$. While

(18) $dLn(g(x)) := |f_0(x), f_1(x), ..., f_n(x); [dLn(g_1(x))|...|dLn(g_n(x))]|$.

Consider the total complex $(Tot^*(B_N^{*,<p}), D)$ of $B_N^{*,<p}$. Then a $(b-1)$-cocycle in the total complex is a sequence $(g, w_1, ..., w_{b-1})$, where $g \in H^\infty(N, AB^{b-1}\mathcal{A}_r^*)$ and $w_j \in S^j_{AB^{b-1-j}\mathcal{A}_r}(N)$ satisfying conditions:

$\sigma(g) = 0$ which means that g is a differentiable mapping from N into $B^{b-1}\mathcal{A}_r$;

$\sigma(w_1) + dLn(g) = 0$ means that w_1 is a connection on the differentiable principal $B^{b-2}\mathcal{A}_r^*$-bundle over N induced by g;

$\sigma(w_{j+1}) + (-1)^j dw_j = 0$ serves as the definition of a $(j+1)$-connection on a differentiable principal $B^{b-2}\mathcal{A}_r^*$-bundle $E \to B$ associated with the mapping g for $1 \le j \le b-2$. Then the sequence $(g, w_1, ..., w_j)$ is called the j-connection bar cocycle.

There exists an equivalence relation in the group of differentiable principal $B^{b-2}\mathcal{A}_r^*$-bundles with $(b-1)$-connections which is induced by the cohomology equivalence relation in the complex $(Tot^*(B_N^{*,<b}), D)$. Thus $\mathsf{H}^{b-1}(Tot^*(B^{*,<b}), D)$ can be identified with a group $\mathsf{E}(N, B^{b-2}\mathcal{A}_r^*, \nabla^{b-1})$ of equivalence classes of differentiable principal $B^{b-2}\mathcal{A}_r^*$-bundles with $(b-1)$-connections.

An assignment $(g, w_1, w_2, ..., w_{b-1}) \mapsto (-1)^{b-1} dw_{b-1}$ induces a homomorphism $K : \mathsf{E}(N, B^{b-2}\mathcal{A}_r^*, \nabla^{b-1}) \to S^b_{\mathcal{A}_r}(N)$ called the curvature of the b-connection $(g, w_1, w_2, ..., w_{b-1})$. The kernel $ker(K)$ is isomorphic to the group $\mathsf{E}(N, B^{b-2}\mathcal{A}_r^*, \nabla^{flat})$ of isomorphism classes of differentiable principal $B^{b-2}\mathcal{A}_r^*$-bundles with flat connections.

51. Curvature of holonomy. If $v, w \in T_0\mathbf{R}^n$, put

(1) $\gamma_{v,w}(u) = 4uv$ for $0 \le u \le 1/4$, $\gamma_{v,w}(u) = v + 4(u-1/4)w$ for $1/4 \le u \le 1/2$, $\gamma_{v,w}(u) = w - 4(u-3/4)v$ for $1/2 \le u \le 3/4$, $\gamma_{v,w}(u) = 4(1-u)w$ for $3/4 \le u \le 1$ and $\gamma^s_{v,w}(u) := \gamma_{sv,sw}(u)$, where $0 \le u, s \le 1$. For a sequence of vectors $\mathbf{w} = (w_0, w_1, ..., w_q)$ in $T_0\mathbf{R}^n$ with $q \in \mathbf{N}$ define a $(q+1)$-dimensional parallelepiped $p[w_0, ..., w_q]$ in the Euclidean space $\mathbf{R^n}$ with $q < n$ if $w_0, ..., w_q$ are linearly independent. Then define $\gamma_{w_0,w_1,w_2}(u_1, u_2) := \gamma_{\gamma_{w_0,w_1}(u_1),w_2}(u_2)$ and by induction

(2) $\gamma_{w_0,...,w_q}(u_1, ..., u_q) = \gamma_{\gamma_{w_0,...,w_{q-1}}(u_1,...,u_{q-1}),w_q}(u_q)$ and $\gamma^s_{\mathbf{w}}(u_1, ..., u_q) := \gamma_{s\mathbf{w}}(u_1, ..., u_q)$, where $0 \le u_1, ..., u_q, s \le 1$. This gives the natural parametrization of the parallelepiped $p[w_0, ..., w_q]$ and the mapping $\gamma_{\mathbf{w}} : \partial I^{q+1} \to \mathbf{R}^n$

which is continuous and piecewise C^∞. Denote by $e_j = (0, ..., 0, 1, 0, ..., 0)$ the standard orthonormal basis in $\mathbf{R}^n$ with 1 in the j-th place. Put $Ln(diag(a_1, ..., a_k)) := diag(Ln(a_1), ..., Ln(a_k))$, where Ln is the principal branch of the logarithmic function with $Ln(1) = 0$ and $diag(a_1, ..., a_n)$ is the diagonal matrix with entries $a_1, ..., a_k \in \mathcal{A}_r^*$.

If h is an $(\mathcal{A}_r^*)^k$-valued C^n holonomy or an homomorphism for a wrap group $(W^M E)_{\infty,H}$ with $\hat{M}$ being H_p^∞ diffeomorphic with ∂I^{m+1} and $\psi = (y_1, ..., y_n)$ is a coordinate system centered at y, $\psi : V \to \mathbf{R}^n$, V is an open neighborhood of a point y in N, then a curvature of h at y is a q-form

(3) $K_y := \sum_{1 \le j_1 < ... < j_q \le n} K_{j_1, j_2, ..., j_q}(y) dy_{j_1} \wedge dy_{j_2} \wedge ... \wedge dy_{j_q} \in \Lambda^q T_y^* N$, where

(4) $K_{j_1,...,j_q}(y) = (-1)^q \lim_{s \to 0} Ln[h(\psi^{-1}(\gamma^s_{e_{j_1},...,e_{j_q}}))] s^{-q-1}$,

where $m \ge q$.

Consider the inversion $(w_j, w_{j+1}) \mapsto (w_{j+1}, w_j)$. In view of Theorem 2 [22] for $\hat{M}$ being H_p^∞ diffeomorphic with ∂I^{m+1} using the iterated loops and the mapping $u_j \mapsto (1 - u_j)$ we get, that

(5) $K_y(w_{g(1)}, ..., w_{g(q+1)}) = (-1)^{|g|} K_y(w_1, ..., w_{q+1})$,

where $g \in S_{q+1}$, S_q denotes the symmetric group of the set $\{1, ..., q\}$, $|g| = 1$ for odd g, while $|g| = 2$ for an even transposition g.

52. Remark. Consider an H^∞ manifold N and a pseudo-manifold X. A mapping $\gamma : X \to N$ is called piecewise C^∞ or H^∞ smooth if it is continuous and the restriction of γ to each top dimensional simplex of X is a C^∞ or H^∞ mapping. A piecewise smooth mapping $\gamma : X \to N$ is called an oriented singular pseudo-manifold q-cycle, if X is an oriented pseudo-manifold q-cycle. Denote by $\mathsf{Z}_q^\psi(N) := \mathsf{Z}_q^\psi(X, N)$ the group of oriented singular pseudo-manifold q-cycles in N.

If there exists an oriented pseudo-manifold with boundary $(Y, \partial Y)$ with a pseudo-diffeomorphism $\eta : \partial Y \to X$ and a piecewise smooth mapping $\zeta : Y \to N$ such that $\gamma = \zeta|_{\partial Y} \circ \eta^{-1}$, where γ is an oriented singular pseudo-manifold q-cycle, then γ is called an oriented singular pseudo-manifold q-boundary in N. Denote by $\mathsf{B}_q^\psi(N) := \mathsf{B}_q^\psi(X, N)$ the group of oriented singular pseudo-manifold q-boundaries in N.

Two oriented singular pseudo-manifold q-cycles $\gamma_j : X_j \to N$, $j = 1, 2$, are homologous, if there exists an oriented $(q+1)$-dimensional pseudo-manifold with boundary $(Y, \partial Y)$ and a piecewise differentiable mapping $\zeta : Y \to N$ such that ∂Y is isomorphic with $X_1 \cup X_2$ and $\zeta|_{X_j} = \gamma_j$ up to an isomorphism $\partial Y \cong X_1 \cup X_2$ for $j = 1, 2$.

Then there exists the group $\mathsf{H}_q^\psi(N) = \mathsf{Z}_q^\psi(N) / \mathsf{B}_q^\psi(N)$ of homology classes of oriented singular pseudo-manifold q-cycles in N, where the group structure is given by the disjoint union.

Consider a twisted $\mathcal{A}_r$ analog of Cheeger-Simons differential group func-

tor consisting of pairs $(h,\alpha) \in Hom(\mathsf{Z}_q^{\psi}(N), \mathcal{A}_r^*) \times \mathsf{S}_{\mathcal{A}_r}^{q+1}(N)_0$ satisfying the condition

(CS) $h(\partial\eta) = \exp((-1)^q \int_\eta \alpha)$ for each $\eta \in \mathsf{S}_{q+1}(N)$,

where $\mathsf{S}_q(N)$ is the group of smooth singular q-chains in N, $\mathsf{S}_{\mathcal{A}_r}^{q+1}(N)_0$ denotes the group of closed differential $\mathcal{A}_r$-valued q-forms on N with $2\pi\mathbf{Z}(\mathsf{C}_r)$-integral periods belonging to $\mathcal{I}_r = \{z \in \mathcal{A}_r : Re(z) = 0\}$, $1 \le r \le 3$. The Cheeger-Simons group $\hat{\mathsf{H}}_\psi^q(N, \mathbf{Z}(\mathcal{C}_r))$ of degree q differential characters on N consists of homomorphisms h described above.

Suppose that X is an H_p^∞ pseudo-manifold. Construct quotients $\mathsf{Z}_q^{\tilde{\psi}}(N)$ and $\mathsf{B}_q^{\tilde{\psi}}(N)$ as quotients of $\mathsf{Z}_q^{\psi}(N)$ and $\mathsf{B}_q^{\psi}(N)$ by the equivalence relation:

$(E1)$ if $\gamma : X \to N$ is an oriented singular pseudo-manifold q-cycle and ξ is a homeomorphism of X such that its restrictions on all top dimensional simplices of a refinement of a triangulation T of X is an H_p^∞ diffeomorphism, then $\gamma \sim \gamma\circ\xi$ and as a class of equivalent elements take $< \gamma >_{\infty,H}$ which is the closure relative to the H_p^∞-uniformity of the family of all such $\gamma\circ\xi$. In view of the Morse and the Sard theorems (see §§II.2.10, 11 [5]) if $\delta \in < \gamma >_{\infty,p}$, then δ is homologous to γ. Put $\mathsf{H}_q^{\tilde{\psi}}(N) := \mathsf{Z}_q^{\tilde{\psi}}(N)/\mathsf{B}_q^{\tilde{\psi}}(N)$, then $\mathsf{H}_q^{\tilde{\psi}}(N) \cong \mathsf{H}_q^{\psi}(N)$ are isomorphic.

53. Higher twisted holonomies. Suppose that $E(N, B\mathcal{A}_r^*, \pi, \Psi)$ is a differentiable principal $B\mathcal{A}_r^*$-bundle with a classifying mapping $g : N \to B^q\mathcal{A}_r^*$ and a q-connection $(g, w_1, ..., w_q)$, where $2 \le r \le 3$. Consider a q-dimensional orientable closed pseudo-manifold X over $\mathcal{A}_r$ and $\gamma : X \to N$ an H^∞ mapping. We have that $B^q\mathcal{A}_r^*$ is q-connected and $g \circ \gamma : X \to B^q\mathcal{A}_r^*$ is homotopic to a constant mapping. This implies an existence of a differentiable mapping $\overline{g\circ\gamma} : X \to AB^{q-1}\mathcal{A}_r^*$ with $\pi \circ \overline{g\circ\gamma} = g \circ \gamma$, where $\pi : AB^{q-1}\mathcal{A}_r^* \to B^q\mathcal{A}_r^*$. On the other hand,

$\pi_*(\gamma^* w_1 + dLn\overline{g\circ\gamma}) = \pi_*\gamma^* w_1 + dLn(g\circ\gamma) = \gamma^*(\pi_* w_1 + dLn(g)) = 0$,

then $(\gamma^* w_1 + dLn(\overline{g\circ\gamma})$ is a $B\mathcal{A}_r$-valued 1-form on X.

The projection $\pi : A\mathcal{A}_r \to B\mathcal{A}_r$ induces the surjective homomorphism $\pi_* : \mathsf{S}_{A\mathcal{A}_r}^j(X) \to \mathsf{S}_{B\mathcal{A}_r}^j(X)$ for each $j = 1, 2, ...$ Therefore, there exists an $A\mathcal{A}_r$-valued 1-form $\bar{w}_j \in \mathsf{S}_{A\mathcal{A}_r}^j(X)$ satisfying the equation:

$\pi_*\bar{w}_1 = \gamma^* w_1 + dLn(\overline{g\circ\gamma})$. Since $\sigma(\gamma^* w_2 - d\bar{w}_1) = \sigma\gamma^* w_2 - d\gamma^* w_1 = \gamma^*(\sigma w_2 - dw_1) = 0$, then $\gamma^* w_2 - dw_1$ is an $\mathcal{A}_r$-valued 2-form on X.

By induction we get, that there exists a differential j-form $\bar{w}_j \in \mathsf{S}_{A\mathcal{A}_r}^j(X)$ such that $\pi_*\bar{w}_j = \gamma^* w_j + (-1)^{j-1} d\gamma^* w_{j-1}$ for each $j = 2, ..., q$. We have that $\sigma(\gamma^* w_j + (-1)^{j-1} d\bar{w}_{j-1}) = \sigma\gamma^* w_j + (-1)^{j-1} d\gamma^* w_{j-1} = \gamma^*(\sigma w_j + (-1)^{j-1} dw_{j-1}) = 0$, consequently, $\gamma^* w_j + (-1)^{j-1} d\bar{w}_{j-1}$ is an $\mathcal{A}_r$-valued j-form on X.

The holonomy of the q-connection $(g, w_1, ..., w_q)$ along $\gamma : X \to N$ is given by

$h(\gamma) = \exp(\int_X (\gamma^* w_q + (-1)^{q-1} d\bar{w}_{q-1}))$.

If there is some other lift $\hat{w}_{j-1}$, then $\hat{w}_{j-1} = \bar{w}_{j-1} + v_{j-1}$, where v_{j-1} is a is an $\mathcal{A}_r$-valued $(j-1)$-form on X. Therefore,

$\int_X (\gamma^* w_j + (-1)^{j-1} d\hat{w}_{j-1}) = \int_X (\gamma^* w_j + (-1)^{j-1} d\bar{w}_{j-1}) + (-1)^{j-1} \int_X dv_{j-1} = \int_X (\gamma^* w_j + (-1)^{j-1} d\bar{w}_{j-1})$ in the considered here case of X with $\partial X = 0$.

This holonomy can be generalized in an abstract way for an equivalence class η of a q-connection $(g, w_1, ..., w_q)$ along a singular oriented pseudo-manifold X of dimension q with an H^∞ mapping $\gamma : X \to N$ such that $h^\eta(\gamma) \in \mathcal{A}_r$. Define $\mathsf{H}^{q+1}(X, \mathbf{Z}(\mathsf{C}_r)(q+1)^\infty_D) := \mathsf{H}^{q+1}(X \setminus S_X, \mathbf{Z}(\mathcal{C}_r)(q+1)^\infty_D)$, where S_X is a singularity of X. If the dimension of X is q, then $\mathsf{H}^{q+1}(X, \mathbf{Z}(\mathcal{C}_r)(q+1)^\infty_D) \cong \mathsf{H}^q(X, \mathcal{A}^*_r)$. Since $codim(S_X) \geq 2$, then $\mathsf{H}^q(X, \mathcal{A}^*_r)$ has a fundamental class that induces an integration along the fundamental class isomorphism and $\mathsf{H}^q(X, \mathcal{A}^*_r) \cong \mathcal{A}^*_r$. Thus we get the isomorphism $T^q_X : \mathsf{H}^{q+1}(X, \mathbf{Z}(\mathsf{C}_r)(q+1)^\infty_D) \to \mathcal{A}^*_r$. Therefore, $h^\eta(\gamma) = T^q_X(\gamma^*(\eta))$ is the holonomy of a q-connection corresponding to an element $\eta \in \mathsf{H}^{q+1}(N, \mathbf{Z}(\mathcal{C}_r)(q+1)^\infty_D)$ along $\gamma : X \to N$ for a singular oriented $\mathcal{A}_r$ pseudo-manifold $\phi : X \to N$ of a real dimension q, where $2 \leq r \leq 3$.

54. Twisted cohomology. Consider a twisted sheaf $\mathcal{B}$ over $\{i_0, ..., i_{2^r-1}\}$. Then a twisted analog of an Alexander-Spanier (or of isomorphic Čech) cohomology with coefficients in $\mathcal{B}$ and supports in the family ϕ is ${}_{AS}\mathsf{H}^*_\phi(X; \mathcal{B}) = \mathsf{H}^*(\Gamma_\phi(\mathcal{S}^* \otimes \mathcal{B}))$.

Take an element $\eta \in {}_{AS}\mathsf{H}^q(\mathcal{S}^{<q+1}_{N,\mathcal{A}_r}(dLn)) := {}_{AS}\mathsf{H}^q(X, \mathcal{S}^{<q+1}_{N,\mathcal{A}_r}(dLn))$, where N is of dimension q. Write it as $\eta = (g_{\mathbf{j}_q}, w_{\mathbf{j}_{q-1}}, ..., w_{\mathbf{j}_0})$, where $\mathbf{j}_b := (j_0, ..., j_b)$ is a multi-index. The $\mathcal{A}^*_r$-valued q-cocycle $g_{\mathbf{j}_q}$ is cohomologous to zero, since N is of dimension q. Therefore, $\mathsf{H}^q(\mathsf{C}^\infty_N(\mathcal{A}^*_r)) \cong \mathsf{H}^{q+1}(N, \mathbf{Z}(\mathsf{C}_r)) \cong 0$. If $\bar{g}_{\mathbf{j}_{q-1}}$ is a $(q-1)$-cochain such that $\delta(\bar{g}_{\mathbf{j}_{q-1}}) = g^{-1}_{\mathbf{j}_q}$, then

$(g_{\mathbf{j}_q}\delta(\bar{g}_{\mathbf{j}_{q-1}}), dLn(\bar{g}_{\mathbf{j}_{q-1}}) + w_{\mathbf{j}_{q-1}}, ..., w_{\mathbf{j}_0}) = (1, dLn(\bar{g}_{\mathbf{j}_{q-1}}) + w_{\mathbf{j}_{q-1}}, ..., w_{\mathbf{j}_0}) =: \eta'$. Denote by D the differential in the twisted complex of $\mathcal{S}^{<q+1}_{N,\mathcal{A}_r}(dLn)$, then the cocycle condition $D(\eta') = 0$ leads to $\delta(dLn(\bar{g}_{\mathbf{j}_{q-1}}) + w_{\mathbf{j}_{q-1}}) = 0$. Since $\mathcal{S}^1_{N,\mathcal{A}_r}$ is an acyclic sheaf, then its twisted complex is exact and inevitably there exists a $(q-2)$-cocycle $\bar{w}_{\mathbf{j}_{q-2}}$ for which $\delta(\bar{w}_{\mathbf{j}_{q-2}}) = dLn(\bar{g}_{\mathbf{j}_{q-1}}) + w_{\mathbf{j}_{q-1}}$. Then $D(1, -\bar{w}_{\mathbf{j}_{q-2}}, 0, ..., 0) + \eta'$ is cohomolous to a cocycle having the form $(1, 0, w'_{\mathbf{j}_{q-1}}, ..., w_{\mathbf{j}_0})$. Continuing this procedure gives a $(q-1)$-cochain $\bar{\eta} = (\bar{g}_{\mathbf{j}_{q-1}}, \bar{w}_{\mathbf{j}_{q-2}}, ..., \bar{w}_{\mathbf{j}_0})$ of the twisted complex $\mathcal{S}^{<q+1}_{N,\mathcal{A}_r}(dLn)$ such that $\eta + D(\bar{\eta}) = (1, 0, ..., 0, \hat{w}_{\mathbf{j}_0})$. From the cocycle condition $D(\eta + D(\bar{\eta})) = 0$ it follows that $\hat{w}_{\mathbf{j}_0}$ is a global q-form on N which we will denote by $\hat{w}$. Put

(1) $T^q_N(\eta) := \exp((-1)^q \int_N \hat{w})$.

The mapping T^q_N of Formula (1) depends only on the cohomology class of η, since if $\eta = D(\bar{\eta})$, then $\hat{w} = 0$. Moreover, T^q_N is independent from a choice of the chain $\bar{\eta}$. Indeed, if $\tilde{\eta}$ is another $(q-1)$-cochain with $\eta + D(\tilde{\eta}) = (1, 0, ..., 0, \nu_{\mathbf{j}_0})$, then $\nu - \hat{w}$ is an $2\pi\mathbf{Z}(\mathsf{C}_r)$-integral q-form and inevitably

$\exp(\int_N(\nu - \hat{w})) = 0$.

Define the isomorphism $t^q_N : \mathsf{H}^q(N, \mathcal{A}^*_r) \to \mathcal{A}^*_r$ as the restriction of T^q_N to $\mathsf{H}^q(N, \mathcal{A}^*_r)$ or it can be written as $t^q_N = T^q_N \circ i^q_N$, where $i^q_N : {}_{AS}\mathsf{H}^q(N, \mathcal{A}^*_r) \to {}_{AS}\mathsf{H}^m(\mathcal{Y}^{<q+1}_{N,\mathcal{A}_r}(dLn))$, $i^q_N(g_{\mathbf{j}_q}) = (g_{\mathbf{j}_q}, 0, ..., 0)$ is the monomorphism induced by the morphism of complexes of sheaves $(\mathcal{A}^*_r)_N \to \mathcal{S}^{<q+1}_{N,\mathcal{A}_r}(dLn)$.

Now construct an isomorphism $e^q_N : \mathsf{H}^q(\mathcal{S}^{<q+1}_{N,\mathcal{A}_r}(dLn)) \to \mathsf{H}^q(N, \mathcal{A}^*_r)$. Consider the q-cocycle η as above, then $dw_{\mathbf{j}_0} = 0$, since N is of dimension q. This implies an existence of a $(q-1)$-cochain $\bar{\eta}$ of the twisted complex of $\mathcal{S}^{<q+1}_{N,\mathcal{A}_r}(dLn)$ such that $\eta + D(\bar{\eta}) = (\bar{g}_{\mathbf{j}_q}, 0, ..., 0)$. The cocycle condition $D(\eta + D(\bar{\eta})) = 0$ implies that $\bar{g}_{\mathbf{j}_q}$ is a locally constant $\mathcal{A}^*_r$-valued cochain. Then the mapping $\eta \mapsto \bar{g}_{\mathbf{j}_q}$ induces the isomorphism $e^q_N : {}_{AS}\mathsf{H}^q(\mathcal{S}^{<q+1}_{N,\mathcal{A}_r}(dLn)) \to {}_{AS}\mathsf{H}^q(N, \mathcal{A}^*_r)$. This construction implies, that e^q_N is the inverse of i^q_N, hence $T^q_N = t^q_N \circ e^q_N$.

55. Theorem. *For an H^∞ manifold N over $\mathcal{A}_r$ the mapping $\eta \mapsto h^\eta$ (see §54) induces an isomorphism $h : \mathsf{H}^q(\mathcal{S}^{<q+1}_{N,\mathcal{A}_r}(dLn)) \to \hat{\mathsf{H}}^q(N, \mathbf{Z}(\mathsf{C}_r))$.*

Proof. It is sufficient to show, that the following diagram with the upper row

$0 \to \mathsf{H}^q(N, \mathcal{A}^*_r) \xrightarrow{i^p_N} \mathsf{H}^q(\mathcal{S}^{<q+1}_{N,\mathcal{A}_r}(dLn)) \xrightarrow{d} \mathsf{S}^{q+1}_{\mathcal{A}_r}(N)_0 \to 0$

and the lower row

$0 \to Hom(\mathsf{H}^\psi_q(N), \mathcal{A}^*_r) \xrightarrow{\hat{i}^p_N} \hat{\mathsf{H}}^q_\psi(N, \mathbf{Z}(\mathsf{C}_r)) \xrightarrow{\pi_2} \mathcal{S}^{q+1}_{\mathcal{A}_r}(N)_0 \to 0$

and with the vertical rows

$\mathsf{H}^q(N, \mathcal{A}^*_r) \xrightarrow{u} Hom(\mathsf{H}^\psi_q(N), \mathcal{A}^*_r)$ and

$\mathsf{H}^q(\mathcal{S}^{<q+1}_{N,\mathcal{A}_r}(dLn)) \xrightarrow{h} \hat{\mathsf{H}}^q_\psi(N, \mathbf{Z}(\mathsf{C}_r))$ commutes. For each $\eta \in \mathsf{H}^q(\mathcal{S}^{<q+1}_{N,\mathcal{A}_r}(dLn))$ put $h(\eta) := (h^\eta, K^\eta)$, where h^η is the holonomy of η and K^η denotes the curvature of η. If $\eta = (g_{j_q}, w_{j_{q-1}}, ..., w_{j_0})$, then $K^\eta = K(g_{j_q}, w_{j_{q-1}}, ..., w_{j_0}) = dw_{j_0}$, consequently, the right hand side of the above diagram commutes.

The universal coefficient theorem isomorphism $u : \mathsf{H}^q(N, \mathcal{A}^*_r) \to Hom(\mathsf{H}^\psi_q(N), \mathcal{A}^*_r)$ is induced by a pairing assigning to a H^∞_p mapping $\gamma : M \to N$ and a cohomology class $\eta \in \mathsf{H}^q(N, \mathcal{A}^*_r)$ an octonion or quaternion number $t^q_M \circ \gamma^*(\eta)$, where t^q_M denotes the restriction of T^q_M to $\mathsf{H}^q(N, \mathcal{A}^*_r)$. Therefore, for $\eta \in \mathsf{H}^q(N, \mathcal{A}^*_r)$ and a q-cycle $\sum_j n_j \gamma_j$, where $\gamma_j : M \to N$ we get $u(\eta)(\sum_j n_j \gamma_j) = t^q_M(\prod_j \gamma^*_j(\eta)^{n_j})$.

From the equalities $h^\eta(\gamma) = u(\eta)(\gamma)$ and $T^q_M = t^q_M \circ e^q_M$ and $e^q_M \circ i^q_M = id$ for an arbitrary H^∞_p mapping $\gamma : M \to N$ it follows that $h^{i^q_N(\eta)}(\gamma) = T^q_M \gamma^* i^q_N(\eta) = T^q_M i^q_M \gamma^*(\eta) = t^q_M e^q_M i^q_M \gamma^*(\eta) = t^q_M \gamma^*(\eta) = \hat{i}^q_N u(\eta)(\gamma)$. Since h is the homomorphism, then the left hand side square of the diagram is commutative as well.

56. Remark. In view of Theorem 55 every element of $\hat{\mathsf{H}}^q_\psi(N, \mathbf{Z}(\mathcal{C}_r))$ is a holonomy homomorphism. The operator T^q_X in the definition of the holon-

omy uses the integration which is invariant under the equivalence relation 52$(E1)$. Then the quotient mapping $\mathsf{Z}_q^{\psi}(N) \to \mathsf{Z}_q^{\tilde{\psi}}(N)$ induces an isomorphism $\hat{\mathsf{H}}^q_{\tilde{\psi}}(N, \mathbf{Z}(\mathsf{C}_r)) \to \hat{\mathsf{H}}^q_{\psi}(N, \mathbf{Z}(\mathsf{C}_r))$, where $\hat{\mathsf{H}}^q_{\tilde{\psi}}(N, \mathbf{Z}(\mathsf{C}_r))$ consists of pairs $(h, v) \in Hom(\mathsf{Z}_q^{\tilde{\psi}}(N), \mathcal{A}_r^*) \times \mathcal{S}^{q+1}_{\mathcal{A}_r}(N)_0$ so that $h(\partial\zeta) = \exp((-1)^q \int_\zeta v)$ for each $\partial\zeta \in \mathsf{B}_q^{\tilde{\psi}}(N)$.

A set theoretic inclusion $H^\infty_p(M, N) \to \mathsf{Z}_q^{\psi}(M, N)$, where q is a dimension of M, induces a group homomorphism $\kappa : (W^M N)_{t,H} \to \mathsf{Z}_q^{\tilde{\psi}}(M, N)$.

Denote by $\mathcal{L}^q_{N,\mathcal{A}_r^*}$ the sheaf associated with the pre-sheaf

$U \mapsto \{\gamma \in Hom^\infty((W^M N)_{\infty,H}, \mathcal{A}_r^*) : supp(\gamma) \subset U\}$. Section 53 and Theorem 55 imply that K^h is an $2\pi\mathbf{Z}(\mathsf{C}_r)$-integral closed $(q+1)$-form on N.

56.1. Lemma. *For each H^∞_p mapping $\zeta : Y \to N$, where $(Y, \partial Y)$ is a pseudo-manifold with boundary ∂Y being a pseudo-manifold over $\mathcal{A}_r$ and for each extension $\hat{h} : \mathsf{Z}_{qb}^{\tilde{\psi}}(M, N) \to G^{kb}$ of an H^∞_p differentiable homomorphism $h : (W^M E)_{b;\infty,H} \to G^{kb}$ being an element of $\hat{\mathsf{H}}^q_{\tilde{\psi}}(N, \mathbf{Z}(\mathcal{C}_r))$ there is the identity:*

$\hat{h}(\partial\zeta) = \exp((-1)^q \int_\zeta K^h)$, where $b \in \mathbf{N}$, $E = E(N, G, \pi, \Psi)$, G is a commutative subgroup in $\mathcal{A}_r^*$, G is isomorphic with $\mathbf{C}^*$.

Proof. In view of the isomorphism $\mathsf{H}^q(\mathcal{S}^{<q+1}_{N,\mathcal{A}_r}(dLn)) \to \hat{\mathsf{H}}^q_{\tilde{\psi}}(N, \mathbf{Z}(\mathcal{C}_r))$ each element of $\hat{\mathsf{H}}^q_{\tilde{\psi}}(N, \mathbf{Z}(\mathcal{C}_r))$ is a holonomy homomorphism. For every pseudo-manifold with boundary $(Y, \partial Y)$ and an H^∞_p mapping $\zeta : Y \to N$ take a partition $\mathcal{T}$ of Y into small cubes Q_j. From the cancellation property of holonomies we get that $\hat{h}(\partial\zeta) = \prod_{Q_j \in \mathcal{T}} \hat{h}(\gamma|_{Q_j})$, since G is commutative. On the other hand, $\hat{h}$ is an extension of h, hence $\prod_{Q_j \in \mathcal{T}} \hat{h}(\gamma|_{Q_j}) = \prod_{Q_j \in \mathcal{T}} h(\gamma|_{Q_j})$. Thus the proof reduces to the case of Y being a $(q+1)$ dimensional cube in $\mathcal{A}_r^m \times \mathbf{R}$ such that ∂Y is embedded into $\mathcal{A}_r^m$ and has the real shadow $\partial[0,1]^{q+1}$.

If μ is a Borel measure on Y relative to which the Sobolev uniformity is given, then $\mu(Y_S) = 0$, since $codim(Y_S) \ge 2$, where S_Y is the singularity of Y. Moreover, a Lebesgue measure on $\mathbf{R}^{q+1}$ induces μ on Y using the fact that $Y \setminus Y_S$ is an $H^{t'}$-manifold with $t' > [(q+1)/2] + 1$.

For matrix-valued over $\mathcal{A}_r$ differential forms $w = (w_{j,k} : j, k = 1, ..., m)$ put $\int_\xi w = (\int_\xi w_{j,k} : j, k = 1, ..., m)$, for diagonal matrices $(a_1, ..., a_m)$ put $\exp(a_1, ..., a_m) := (e^{a_1}, ..., e^{a_m})$, if $a_1 \ne 0$,...,$a_m \ne 0$, then $Ln(a_1, ..., a_m) = (Ln(a_1), ..., Ln(a_m))$.

Without loss of generality h is additive and $\mathbf{R}$ homogeneous on $\mathsf{Z}_q(N)$. For each $n \in \mathbf{N}$ divide $[0,1]$ into n small subintervals, that induces a subdivision of $[0,1]^{q+1}$ into n^{q+1} cubes with vertices denoted by $v_{j_1,...,j_{q+1}}(n)$, where $j_1, ..., j_{q+1} = 0, 1, ..., n$. Consider the wrap $\gamma^n_{j_1,...,j_{q+1}} := \gamma_{e_1/n,...,e_{q+1}/n} +$

$v_{j_1,...,j_{q+1}}(n)$, where $e_1,...,e_{q+1}$ is the standard basis of $\mathbf{R}^{q+1}$.

Take $\xi \in H^\infty_p(Y,E)$ such that $\pi \circ \xi = \gamma$. Therefore,

$\int_\xi K = \int_Y \xi^*K = \lim_{n\to\infty} \sum_{j_1,...,j_{q+1}} \xi^*K(v_{j_1,...,j_{q+1}}(n))n^{-q-1}$
$= (-1)^q \lim_{n\to\infty} \sum_{j_1,...,j_{q+1}} \lim_{s\to 0}[Ln\ h(\xi \circ \gamma^n_{j_1,...,j_{q+1}})]s^{-q-1}n^{-q-1}$,
where $\xi^*K_y = \xi^*K(y)dx_1 \wedge ... \wedge dx_{q+1}$ for each $y \in N$. Taking $s = 1/n$ gives

$\lim_{n\to\infty} \lim_{s\to 0} \sum_{j_1,...,j_{q+1}}[Ln\ h(\xi \circ \gamma^n_{j_1,...,j_{q+1}})]s^{-q-1}n^{-q-1}$
$= \lim_{n\to\infty} \sum_{j_1,...,j_{q+1}} Ln\ h(\xi \circ \gamma^n_{j_1,...,j_{q+1}})$
$= \lim_{n\to\infty} Ln(\prod_{j_1,...,j_{q+1}} h(\xi\circ\gamma^n_{j_1,...,j_{q+1}})) = \lim_{n\to\infty} Ln\ h(\sum_{j_1,...,j_{q+1}} \xi\circ\gamma^n_{j_1,...,j_{q+1}})$
$= \lim_{n\to\infty} Lnh(\xi \circ \gamma^n_{0,...,0}) = Ln\ h(\gamma)$,
since $h(\gamma_1\lambda\lambda^{-1}\gamma_2) = h(\gamma_1\gamma_2)$ and G is commutative, where $\lambda : Y \to N$ is a path joining marked points y_1 and y_2 of wraps γ_1 and γ_2, that is $\gamma_j(\hat{s}_{0,q}) = y_j$ and $\lambda(\hat{s}_{0,q}) = y_1$, $\lambda(\hat{s}_{0,q+k}) = y_2$ while $\lambda^{-1}(\hat{s}_{0,q}) = y_2$ and $\lambda^{-1}(\hat{s}_{0,q+k}) = y_1$ for each $j = 1, 2$ and $q = 1,...,k$.

57. Lemma. *Suppose that $\phi : A \subset X$ is a pointed inclusion of CW-complexes and $\theta : X \to X/A$ is the quotient mapping. Let a group G be twisted over $\{i_0,...,i_{2^r-1}\}$. Then $\theta_* : (W^ME; X, G, \mathbf{P})_{t,H} \to (W^ME; X/A, G, \mathbf{P})_{t,H}$ is a principal $(W^ME; A, G, \mathbf{P})_{t,H}$-bundle.*

Proof. Let G, E and B be topological groups so that G acts effectively on E. Consider U open in B with $e \in U$. Suppose that $\pi : E \to B$ is an open surjective mapping. Each G-equivariant mapping $\xi : \pi^{-1} \to G$ induces a local trivialization of $\pi : E \to B$ over U. A group structure in E induces a system of local trivializations of E/B. It is described as follows. For each $v \in E$ take an open subset $U_v = \pi(v\pi^{-1}(U))$ in B. Then the family $\{U_v : v \in E\}$ forms an open covering of B. For each $v \in E$ there exists a G-equivariant mapping $\xi_v : \pi^{-1}(U_v) = v\pi^{-1}(U) \to G$ given by $\xi_v(x) = \xi(v^{-1}x)$.

Therefore, an open surjective mapping $\pi : E \to B$ is a principal G-bundle if and only if there exists a neighborhood U of the unit element e in B and a G-equivariant mapping $\xi : \pi^{-1}(U) \to G$.

Since the group G is twisted, then due to Proposition 19 and Theorem 20 [22] it is sufficient to prove this Lemma for the commutative group G_0.

Consider a deformation retraction $\eta : [0,1] \times V \to A$ of V onto A, where V is an open neighborhood of A, put $U = \theta_*[(W^ME; V, G_0, \mathbf{P})_{t,H}]$. A $(W^ME; A, G_0, \mathbf{P})_{t,H}$-equivariant mapping $\xi : (\theta_*)^{-1}(U) = (W^ME; W, G_0, \mathbf{P})_{t,H} \to (W^ME; A, G_0, \mathbf{P})_{t,H}$ is given by the formula $\xi(< \mathbf{P}_{\hat{\gamma},v} >_{t,H}) =< \eta(1, \mathbf{P}_{\hat{\gamma},v} >_{t,H}$ due to Propositions 7.1 and 13(2) [22].

58. Theorem. *For each connected smooth manifold N, the homomorphism κ induces an isomorphism*

$\kappa_* : \pi_0((W^ME; N, \mathcal{A}^*_r, \mathbf{P})_{b;\infty,H}) \to \mathsf{H}^{\tilde{\psi}}_{qb}(N, \mathbf{Z}(\mathsf{C}_r))$, *where $1 \le b \in \mathbf{N}$, q is a dimension of M.*

Proof. The uniform space $H^\infty_p(M,E)$ is everywhere dense in the uni-

form space $C^0(M,E)$ of all continuous mappings from M into E, where M is an H^∞_p-pseudo-manifold. Therefore, there exists an extension of $N \mapsto \pi_0((W^ME;N,\mathcal{A}_r^*,\mathbf{P})_{b;\infty,H})$ to a functor on the category of pointed CW-complexes and pointed continuous mappings, that does not change a homotopy type.

Recall a reduced homology theory. It is a functor H_* from the category of pointed CW-complexes and pointed continuous mappings into the category of graded twisted groups satisfying the properties $(H1-H4)$.

$(H1)$. For each pointed continuous mapping of CW-complexes $f: X \to Y$ and $a \in \mathbf{Z}$, the induced homomorphism $f_* : \mathsf{H}_a(X) \to \mathsf{H}_a(Y)$ depends only on the homotopy type of f.

$(H2)$. For each pointed CW-complex X and $a \in \mathbf{Z}$ there is a natural isomorphism

$\Sigma_X : \mathsf{H}_a(X) \to \mathsf{H}_{a+1}(\Sigma X)$,

where ΣX is a reduced suspension of X.

$(H3)$. For each pointed inclusion $i: A \subset X$ of CW-complexes and $a \in \mathbf{Z}$ the sequence

$\mathsf{H}_a(A) \xrightarrow{i_*} \mathsf{H}_a(X) \xrightarrow{g_*} \mathsf{H}_a(X/A)$ is exact,

where $g: X \to X/A$ is the quotient mapping.

$(H4)$. $\mathsf{H}_a(S^1) = e$ for $a \ne 1$ and $\mathsf{H}_1(S^1) = \mathbf{Z}$. These properties are standard and they are demonstrated in Lemma 4.5 [12] for commutative groups. Due to Conditions $4(A1,A2)$ on twisted groups we get the reduced twisted homology theory.

In view of Lemma 57 $\pi_j((W^ME;A,G,\mathbf{P})_{t,H}) \to \pi_j((W^ME;X,G,\mathbf{P})_{t,H}) \to \pi_j((W^ME;X/A,G,\mathbf{P})_{t,H})$ is a fragment of the long exact homotopy sequence of the fibration θ_*, where G is the twisted group over $\{i_0,...,i_{2^r-1}\}$, $j = 0,1,2,....$ Moreover, Conditions $(H2,H3)$ follow from Lemma 57. Therefore, Properties $(H1-H4)$ for twisted groups are direct consequences of the corresponding properties for commutative groups. Though for the proof of this theorem the case of commutative graded groups is sufficient.

Since κ is a natural transformation of homology theory and in view of Proposition 19 and Theorem 20 [22] this induces the isomorphism κ^*.

59. Proposition. *The curvature morphism* $K : \mathcal{L}^q_{N,\mathcal{A}_r^*} \to \mathcal{S}^{q+1,cl}_{N,\mathcal{A}_r}$ *is an isomorphism.*

Proof. The family $\mathbf{C}_M := \mathbf{R} \oplus M\mathbf{R}$ with $M \in \mathcal{A}_r$, $Re(M) = 0$ and $|M| = 1$ is such that its union gives $\bigcup_M \mathbf{C}_M = \mathcal{A}_r$. In view of Theorem 55 and Lemma 56.1 K is a monomorphism and an epimorphism from $\mathcal{L}^q_{N,\mathcal{A}_r^*}$ onto $\mathcal{S}^{q+1,cl}_{N,\mathcal{A}_r}$. This gives the statement of this proposition.

60. Theorem. *The restriction homomorphism* $\kappa^* : Hom(\mathsf{Z}^{\tilde{\psi}}_{qb}(M,N),\mathcal{A}_r^*) \to Hom((W^MN)_{b;\infty,H},\mathcal{A}_r^*)$ *induces an isomorphism*

$\hat{\kappa}: \hat{\mathsf{H}}^{qb}_{\tilde{\psi}}(N, \mathbf{Z}(\mathsf{C}_r)) \to Hom^\infty((W^M N)_{b;\infty,H}, \mathcal{A}_r)$, *where* $1 \le b \in \mathbf{N}$, q *is a covering dimension of* M, M *and* N *are over* $\mathcal{A}_r$.

Proof. Each homomorphism $h : \mathsf{Z}^{\tilde{\psi}}_{qb}(M, N) \to \mathcal{A}^*_r$ is a holonomy of $\hat{\mathsf{H}}^{qb}_{\tilde{\psi}}(N, \mathbf{Z}(\mathsf{C}_r))$, since the holonomy induces an isomorphism $\mathsf{H}^{qb}(\mathcal{S}^{<qb+1}_{N,\mathcal{A}_r}(dLn)) \to \hat{\mathsf{H}}^{qb}_{\tilde{\psi}}(N, \mathbf{Z}(\mathsf{C}_r))$. Then the restriction of h on $(W^M E)_{b;\infty,H}$ is of H^∞ class, where $E = E(N, G, \pi, \Psi)$ with $G = \mathbf{C}_M$ (see §56.1). Therefore, $\kappa^*(h) \in Hom^\infty((W^M N)_{b;\infty,H}, \mathcal{A}^*_r)$, since $(W^M E)_{b;\infty,H}$ is the principal G^{bk}-bundle over $(W^M N)_{b;\infty,H}$.

Consider an extension $\hat{h} : \mathsf{Z}^{\tilde{\psi}}_{qb}(M, N) \to \mathcal{A}^*_r$ of h, hence $\hat{h} \in \hat{\mathsf{H}}^{qb}_{\tilde{\psi}}(N, \mathbf{Z}(\mathsf{C}_r))$.

We have the locally analytic mapping Ln from $\mathcal{A}^*_r$ onto $\mathcal{A}_r$. The group $(W^M N)_{b;\infty,H}$ is commutative, therefore instead of $Hom^\infty((W^M N)_{b;\infty,H}, \mathcal{A}^*_r)$ we can consider the commutative additive group $Hom^\infty((W^M N)_{b;\infty,H}, \mathcal{A}_r)$, where $\mathcal{A}_r$ is considered as the additive group $(\mathcal{A}_r, +)$. At the same time the group $\mathbf{Z}(\mathcal{C}_r)$ is commutative. Then $Ln(\kappa^*(h)) \in Hom^\infty((W^M N)_{b;\infty,H}, \mathcal{A}_r)$.

For each $\xi \in \mathsf{Z}^{\tilde{\psi}}_{qb}(M, N)$ there exists $\zeta \in (W^M N)_{b;\infty,H}$ and $\partial\eta \in \mathsf{B}^{\tilde{\psi}}_{qb}(M, N)$ such that $\xi = \kappa(\zeta) + \partial\eta$, since $\kappa_* : \pi_0((W^M E; N, \mathcal{A}^*_r, \mathbf{P})_{b;\infty,H}) \to \hat{\mathsf{H}}^{qb}_{\tilde{\psi}}(N, \mathbf{Z}(\mathsf{C}_r))$ is an isomorphism due to Theorem 58. Then for each extension $\hat{h} : \mathsf{Z}^{\tilde{\psi}}_{qb}(M, N) \to \mathcal{A}^*_r$ of h there is the identity:

$\hat{h}(\xi) = h(\zeta) + \exp((-1)^{qb} \int_\eta K^h)$

due to Section 53 and Lemma 59. Therefore, h has a unique extension $\hat{h}$. This implies that $\hat{\kappa}$ is an isomorphism.

61. Remark. Mention that Theorems 55, 58 and 60 can be proved in another way using the corresponding statements over $\mathbf{C}$ and the twisted structure of sheaves over $\{i_0, ..., i_{2^r-1}\}$.

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